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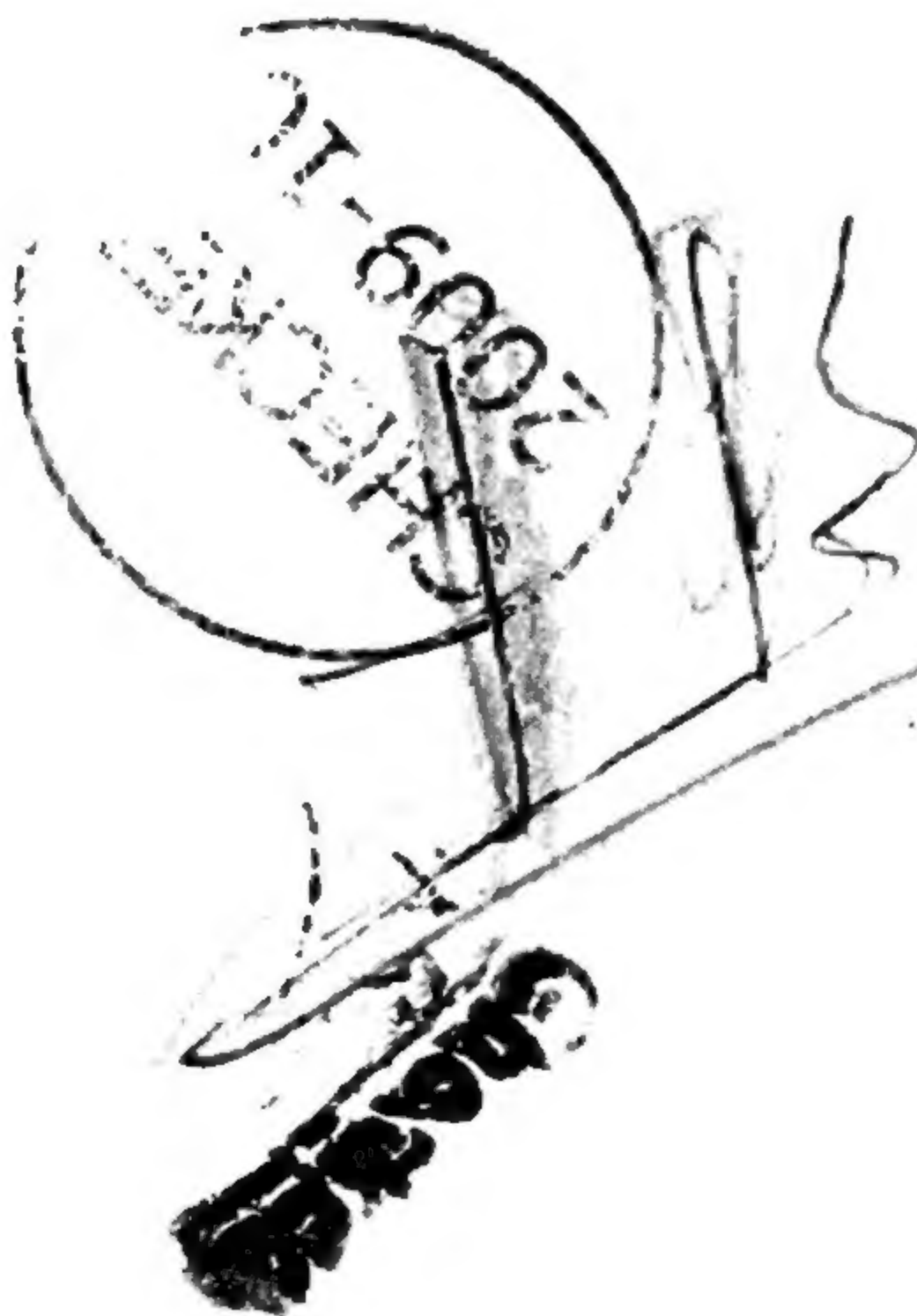
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MODERN PURE GEOMETRY

Pure Geometry



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A TEXT BOOK
OF
Modern Pure Geometry

(For B.A. students of the Panjab University)

BY

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PREFACE.

This book is intended as a text book on Pure Geometry for those B.A. students of the Panjab University who take up Mathematics B Course as one of their optional subjects. The syllabus followed is the one laid down by the board of studies in Mathematics in the year 1943.

Hints to most of the exercises have been given at the end of the book in order to encourage the average student to take interest in them. It is hoped that he will make a wise use of these hints. The solutions suggested by these hints may not necessarily be the simplest and it is possible that independent efforts lead to simpler solutions in some cases.

The Authors shall be very grateful for any suggestions for improvements or correction of text or exercises.

LAHORE,
Nov. 1943.

Authors.

SYLLABUS.

Euler's theorem on four collinear points. Ceva's and Menelau's theorems expressed as ratios of segments with due regard to their signs.

Projection.—A range of three points is projective with any other range of three points in space. The projection of a straight line to infinity and at the same time two angles into given angles.

Desargue's theorem on triangles in perspective (in plane and space).

Harmonic properties of elementary forms; complete quadrilateral and quadrangle.

Cross-ratio of elementary forms viz. range of points and pencils of lines. Twenty-four cross-ratios are equivalent to six, all of which can be expressed in terms of any one of them. Cross-ratio is unaltered by projection. If $A B C D$, $A' B' C' D'$ are two distinct ranges of points such that $(A B C D) = (A' B' C' D')$, then the ranges are projective.

Two homographic pencils are mutually projective.

Involution as $(1, 1)$ rational, algebraic, reversible correspondence between two ranges of points (or pencil of lines). The necessary and sufficient condition that three pairs of elements may belong to an involution. A range (pencil) in involution projects into a range (pencil) in involution. In every involution pencil, there is only one pair of orthogonal rays, unless the involution pencil is orthogonal. Every involution pencil can be projected into an orthogonal involution pencil. Radical axis, coaxial circles, limiting points, Salmon's theorem, Pole and polar properties of circle and conic.

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CHAPTER I

INTRODUCTION

1.1. **Figures. Elements.** By a geometrical figure is meant any assemblage of points, *lines, and planes; the points, lines, and planes, which constitute a figure, are known as the elements of the figure.

1.2. Properties of geometrical figures divide themselves into two classes: (i) **Metrical**, i.e., those properties which deal with the **magnitude** of figures and thus involve the idea of *measurement* of lengths, angles, etc., and (ii) **Descriptive**, i.e., those properties which deal with the relative **positions** of the elements (points, lines, planes) constituting a figure apart from any consideration of the magnitudes; e.g., 'The angle in a semi-circle is a right angle,' is a metrical property whereas the property 'A line meets a circle in two points' is a descriptive one. In this book, we shall be mostly concerned with Descriptive geometry which deals with the descriptive properties of geometrical figures.

1.3. The following fundamental relations between the elements are assumed:

(i) One and only one straight line can be drawn through two points.

(ii) Two co-planar straight lines must intersect in a finite point or be parallel.

(iii) A plane is determined by (a) two intersecting straight lines, or (b) a straight line and a point outside it, or (c) three non-collinear points.

*By 'line', we shall always understand a straight line unless the contrary is mentioned.

(ii) A straight line which does not wholly lie on a plane meets the plane in one point only or is parallel to it.

(v) Two planes intersect in a straight line or are parallel.

(vi) Through a given point, one and only one straight line can be drawn parallel to a given line.

The lines and planes are all to be considered as extending beyond limits. When we say that two elements are parallel, we mean that the elements concerned do not meet, however far they may be produced.

1.31. Incidence of two elements. When a point lies on a line, or in a plane, it is said to be '*incident*' to the line or the plane. Similarly, when a line passes through a point, or lies in a plane, it is said to be '*incident*' to the point or the plane. Finally, when a plane passes through a point or a line, it is said to be '*incident*' to the point or the line. Thus

Two elements are said to be incident when one lies on, or passes through, the other.

1.4. Notation. The study of the subject is greatly facilitated by the systematic use of the following notation :—

1. Points are denoted by italic capitals, A , B , C , D etc.

2. Lines are denoted by small italics, a , b , c , d , etc.

3. Planes are denoted by small Greek letters π , α , β , γ , etc.

4. The symbol AB denotes the straight line passing through the two points A , B .

5. The symbol (ab) denotes the point of intersection of the two coplanar lines a , b .

6. The symbol Aa or aA denotes the plane through the point A and the line a .

7. The symbol ABC denotes the plane through three non-collinear points A, B, C .

8. The symbol $a\alpha$ denotes the point where the line a meets the plane α .

The essential background of the notation explained above is the notion of incidence, so that a combination of symbols denoting different elements denotes the element which is incident to the different elements. Thus the symbol $a\alpha$ denotes the element to which both, the line a and the plane α , are incident, viz., the point of intersection of the line and the plane.

Note. The triangle formed by three points A, B, C is also denoted by the symbol ABC .

Ex. What do the symbols $\alpha.BC$, $A.\beta\gamma$, $\alpha.Aa$, $A.\mathcal{T}a$ denote.

1.5. Elementary Geometric Forms. A group of like geometric elements is said to be a **form**, e.g., points on a line, lines through a point, lines in a plane etc., are different forms. We give below the definitions of what are called **Elementary Forms**.

(i) A set of points lying on a line is called a **Range**. The line of collinearity is called the base of the range.

(ii) A set of coplanar lines through a point is called a **Flat Pencil** or merely **Pencil**. The point of concurrence is called the vertex of the Pencil.

(iii) A set of planes through a line is called an **Axial Pencil**. The planes are said to be **co-axial** and the line is called the **axis** of the set.

(iv) The aggregate of points in a plane is called the field of points. The plane is called the base.

(v) The aggregate of lines in a plane is called the field of lines. The plane is called the base.

(vi) A set of concurrent lines in space is called a bundle of lines, the point of concurrence being the **base**.

(vii) A set of planes through a point is called a **bundle of planes**, the common point of the set being called the **base**.

We observe that the elements of every form are associated with a common element not belonging to the form. This common element has been called the **base of the form**, e.g., all the points which constitute a *field of points* lie on the same plane so that the plane is the base of this form.

1.51. Dimensions of Forms. The first three forms, viz., the range, the flat pencil and the axial pencil are termed as *one-dimensional* geometric forms, and the last four forms, viz., the field of points, the field of lines, the bundle of lines, and the bundle of planes, each of which possesses the property of including an infinite number of one-dimensional forms are termed as *two-dimensional* geometric forms.

The student acquainted with *Analytical Geometry* knows that the analytical expressions for the first three forms involve only *one arbitrary parameter*, whereas the analytical expressions for the last four involve *two arbitrary parameters*. The analytical expressions for the various forms are given here below :—

(i) If (x_1, y_1, z_1) and (x_2, y_2, z_2) be two points of a range, then the range is given by the set of points

$$\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}, \frac{z_1 + \lambda z_2}{1 + \lambda} \right)$$

where λ is the arbitrary parameter so that by varying λ , we get different points of the range.

(ii) If $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ be two rays of a flat pencil, then the pencil is given by the set of lines

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0,$$

λ being the parameter.

(iii) If $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ be two planes of an axial pencil, then the pencil is given by the set of planes

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0 \quad (\lambda, \text{parameter})$$

(iv) By selecting two arbitrary lines on a plane as co-ordinate

axes, any point on the plane is determined by its two co-ordinates (x, y) so that a field of points is the set of points (x, y) where x and y are the two independent parameters.

(v) A field of lines is given by the set of equations

$$ax + by + c = 0,$$

the two ratios $a : b : c$ being the two independent parameters.

(vi) A bundle of lines through any point (α, β, γ) is given by the set of equations

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

the two ratios $l : m : n$ being the two independent parameters.

(vii) A bundle of planes through any point (α, β, γ) is given by the set of equations

$$A(x - \alpha) + B(y - \beta) + C(z - \gamma) = 0,$$

the two ratios $A : B : C$, being the two independent parameters.

1.6. One-to-one correspondence. Consider a class of students and their roll numbers. Corresponding to each student, there is one roll number and corresponding to a given roll number, there is only one student so that we say that the students and their roll numbers are in one-to-one correspondence. Again, consider the class of married women and the class of married men in a country where polygamy and polyandry are prohibited. To each married woman there corresponds one married man, and to each married man there corresponds one married woman so that we say that each class is in one-to-one correspondence with the other. Thus

Two sets of different objects are said to be in one-to-one correspondence with each other, when to each member of either set, there corresponds one and only one member of the other set.

One-to-one correspondence is also sometimes expressed as $(1-1)$ correspondence.

Ex. Show that the aggregate of positive integers is in $(1-1)$ correspondence with the aggregates of odd and even positive integers.

161. One-to-one correspondence between a range of points and a pencil of lines. Consider a pencil of lines and a range of points in the plane of the pencil. The totality of the rays of the pencil (with one exception considered in §17) and the totality of the points of the range can be put into one-to-one correspondence thus:— To any point A of the range, we associate the ray a of the pencil joining A to the vertex V ; and to any ray a of the pencil, we associate the point A of the range, where the ray intersects the base x of the range.

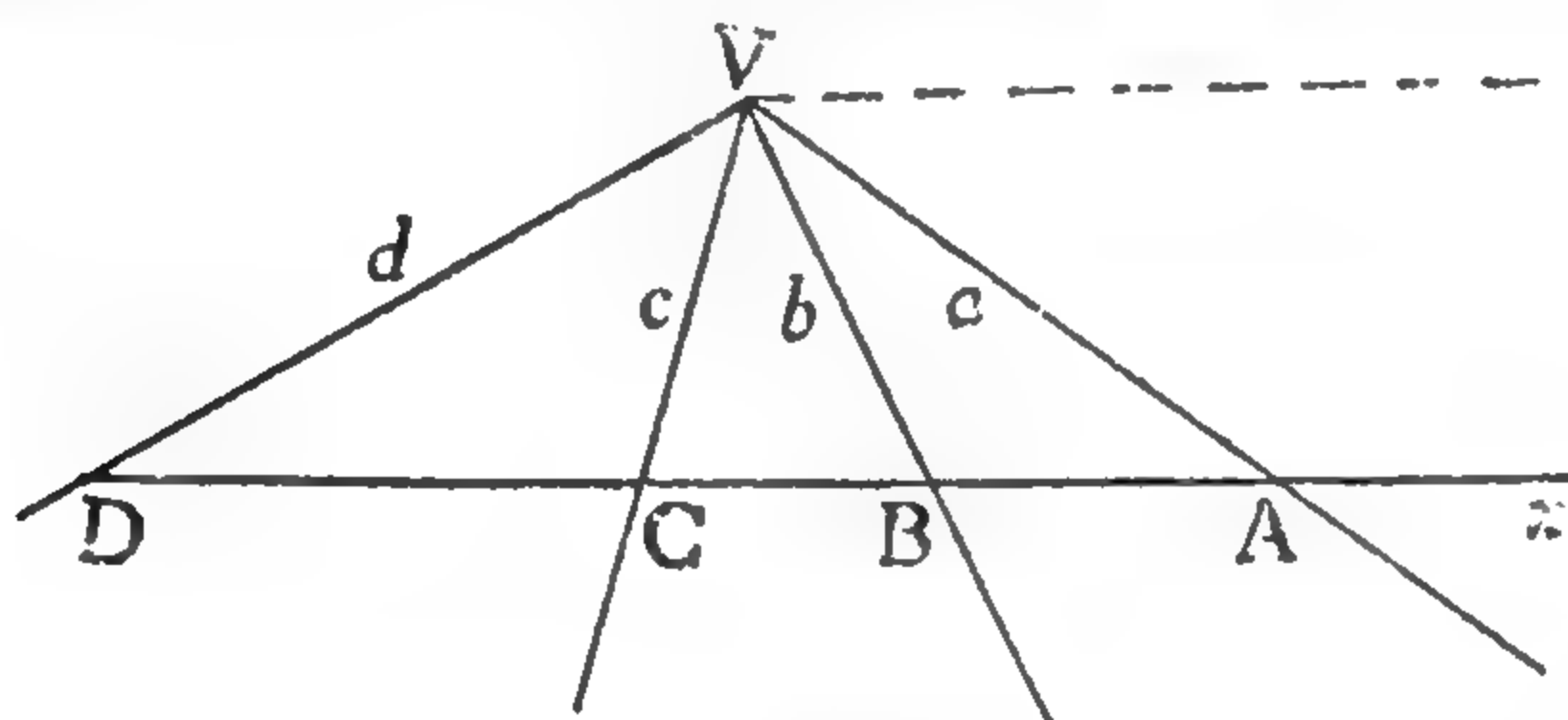


Fig. 1.

Similarly we can put a pencil of lines in one-to-one correspondence with a co-axial pencil of planes.

17. Elements at infinity. We have seen that a one-to-one correspondence can be set up between a pencil of lines and a range of points lying in the plane of the pencil. There is one notable exception, however, which we shall consider now. No point of the range corresponds to the ray of the pencil parallel to the base x of the range, because the point which corresponds to a ray is the point of intersection of the ray and the base x and this particular ray, being parallel to x , does not meet the base x . In order to meet this exception, we assume the existence of an ideal point on the base x , called the 'point at infinity' on the base x , and say that this point at infinity on the base x corresponds to the ray which is parallel to the base x . Since one and only one line can be drawn through a given point parallel to a given line, we assume the existence of *only one* such ideal point on the line so as to complete the framework

of one-to-one correspondence between the elements of the two forms. This point can be taken to lie at either end of the line.

Thus we have assumed that *every straight line has one point at infinity on it and that two parallel straight lines have the same point at infinity.*

With this *fundamental* assumption, our statements about parallelism undergo a notable transformation. Instead of saying that two parallel lines do not meet, we shall say that they 'meet at infinity' or have a 'common point at infinity.' Conversely, two lines through a point at infinity are parallel. It also follows that a system of parallel lines can be looked upon as concurrent lines, the point of concurrence being the point at infinity which is common to all the lines.

1.71. Line at infinity. Since every line on a given plane possesses a point at infinity, and two parallel lines have the same point at infinity, therefore by varying the directions of the lines, we will obtain different points at infinity lying on the plane. The locus of these points at infinity lying on the plane which is such that every line on the plane meets it in one and only one point is considered to be a line, called the **line at infinity** on the plane.

It is easy to prove that *two parallel planes α and β have the same line at infinity.* This will be proved if we show that a point at infinity on the plane α is also a point at infinity on β and conversely. Let P be a point at infinity on α , and let a be any line on α drawn through P . Any plane through the line a meets the plane β in a line b parallel to a , since any plane cuts a pair of parallel planes in a pair of parallel lines. Since the line b is parallel to the line a , the point at infinity P on a also lies on b and, therefore on β .

Hence the result.

Thus a set of parallel planes can be considered as co-axal, the axis being the common line at infinity of the system of planes.

As above, it can also be shown that a line parallel to a plane meets the plane in a point at infinity.

1.72. Plane at infinity. The locus of the lines at infinity which is such that every plane meets it in a line is considered to be a plane, called the **Plane at infinity**.

1.73. The facility of argument attained with the help of the elements at infinity is illustrated by the following example. To prove that if b and c are $\parallel a$, b is also $\parallel c$, we observe that

$\therefore$ b is $\parallel a$, the point at infinity on b coincides with the point at infinity on a .

Similarly, the point at infinity on c coincides with the point at infinity on a .

$\therefore$ b and c have a common point at infinity and hence must be parallel.

Ex. 1. Give two examples of uniformity of statement attained by the introduction of the elements at infinity.

Ex. 2. If a and b are each parallel to α , must a be $\parallel b$?

Ex. 3. Two planes are drawn one through each of two parallel straight lines. Show that the planes are parallel or their line of section is parallel to the given line.

1.8. Duality. A plane geometrical figure is an assemblage of points and lines. We can look upon it as having been *primarily* described by a point, the straight lines (each of which is the join of two points) being incidental to such a description or, it can be considered to have been primarily made up of straight lines, the points appearing incidentally as the intersections of the lines. In the first case, the point is taken as the *primary* element and the line becomes the *secondary* element. In the second case, the line is the primary element and the point secondary.

A curve, with the point as our primary element, is called the *locus* (Fig. 2) of the point, and with the line as

our primary element, the *envelope* (Fig. 3) of the moving line.

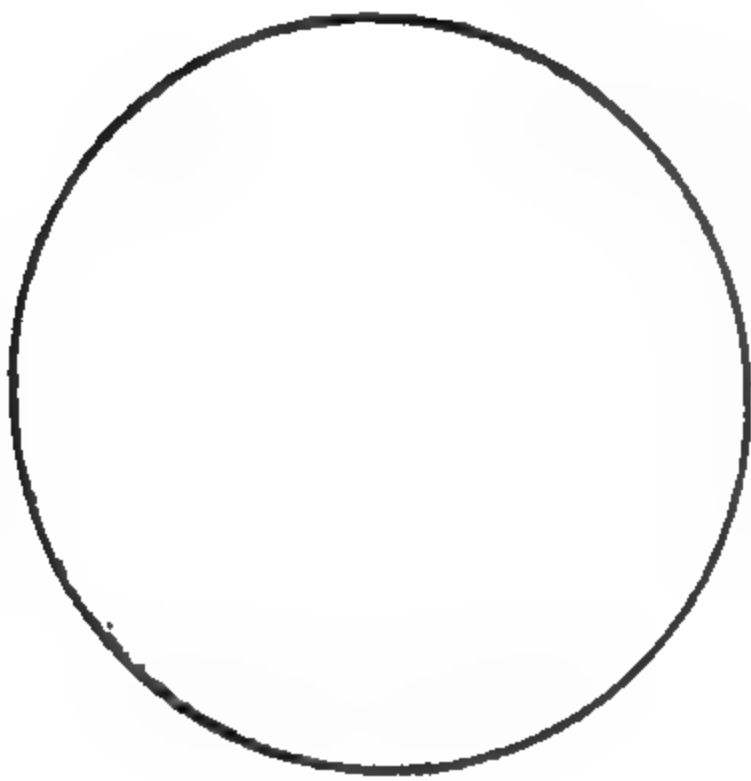


Fig. 2.

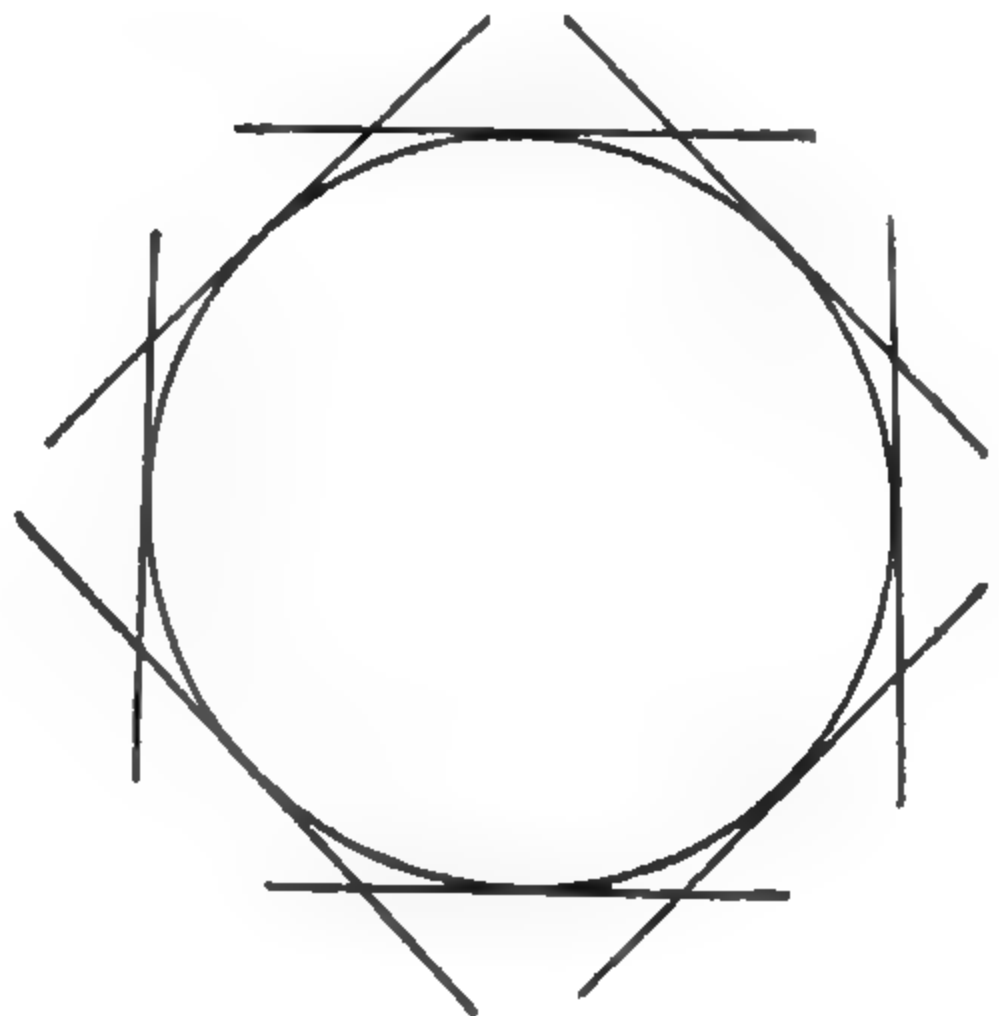


Fig. 3.

Thus a geometrical figure is capable of a *dual* description according as the point or the line is taken as our primary element.

This will also be apparant from the consideration of two very simple cases given below. The diagrams are drawn side by side to help the student to a clear understanding of the point.

1·81. Three non-collinear points give a triangle ABC , the three sides being the lines joining the three points in pairs.

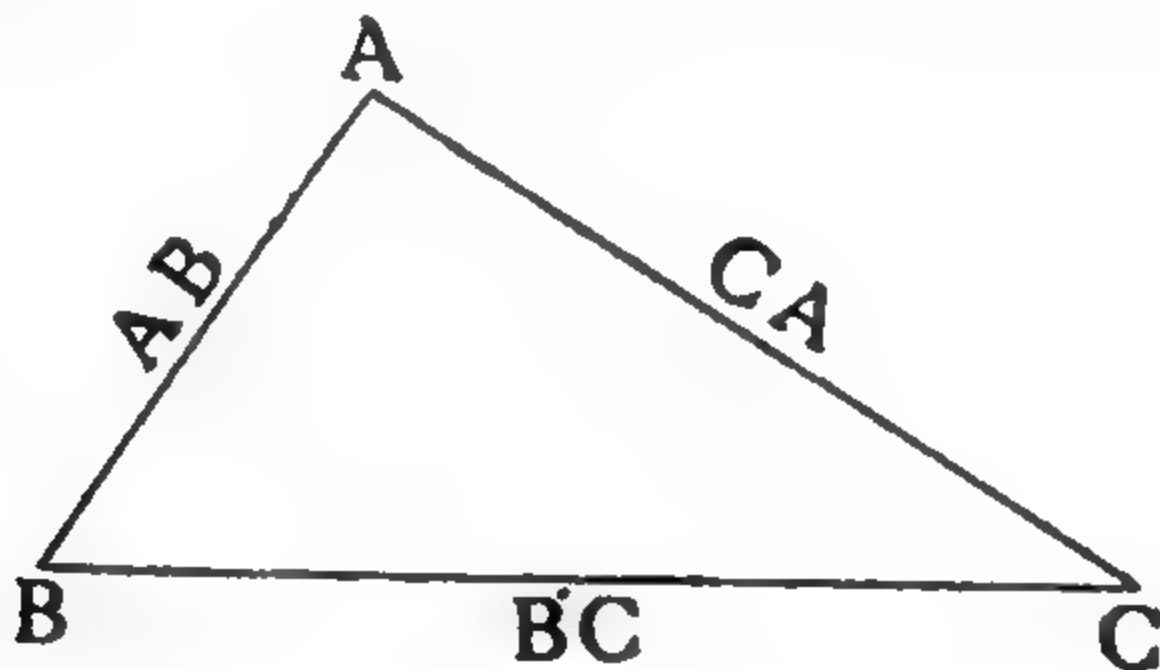


Fig. 4.

1·82. Three non-concurrent lines give a triangle, the three vertices being the points of intersection of the three lines taken in pairs.

Here to the points A, B, C of the first triangle correspond the sides a, b, c of the second and to the sides BC, CA, AB of the first correspond the points $(bc), (ca), (ab)$ of the second. It will be observed that to each point in one figure, there corresponds only one side of the second.

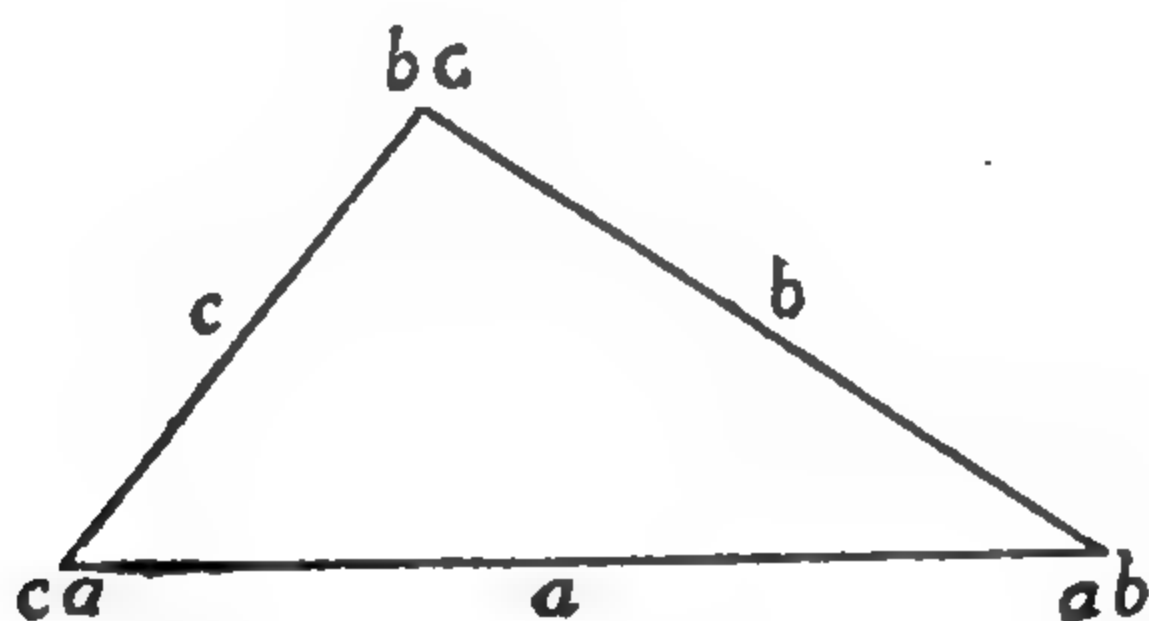


Fig. 5.

1183. The complete quadrangle and the complete quadrilateral.

Four points of a plane no three of which are collinear constitute a quadrangle.

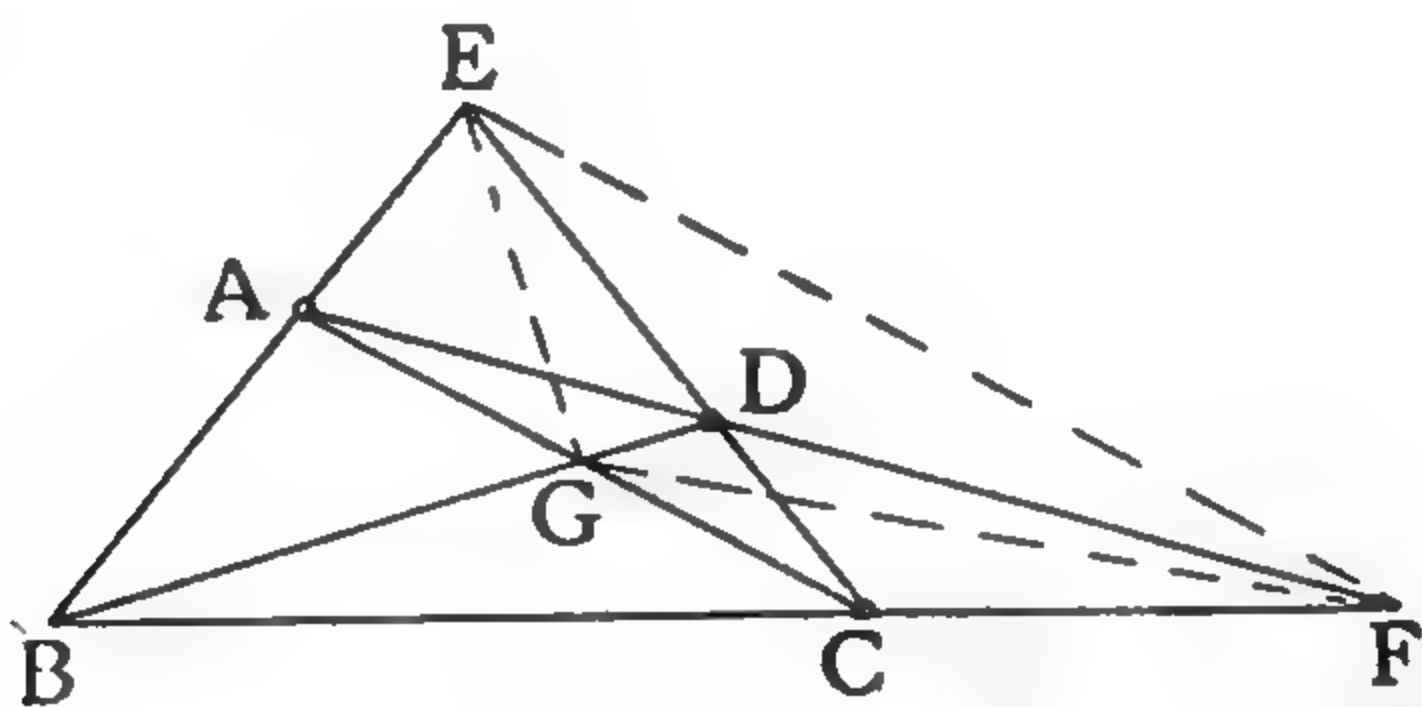


Fig. 6.

The four points can be joined in pairs with the help of six straight lines, called the sides of a quadrangle.

The figure so obtained is called a **complete quadrangle**.

Two sides are said to be opposite, if, when one passes through two of the points, the other passes through the remaining two. The three pairs of opposite sides in (fig. 6) are AB, CD ; AC, BD ; AD, BC .

The points of intersection E, F, G of the pairs of opposite sides are called **Diagonal Points**, and the triangle EFG is called the **diagonal-point triangle** of the quadrangle.

It will be seen that through every point of the quadrangle there pass three sides.

Ex. Show that the centres of the inscribed and escribed circles of a given triangle form a complete quadrangle of which the given triangle is the diagonal-point triangle.

184. *Four lines in a plane no three of which are concurrent form a quadrilateral.*

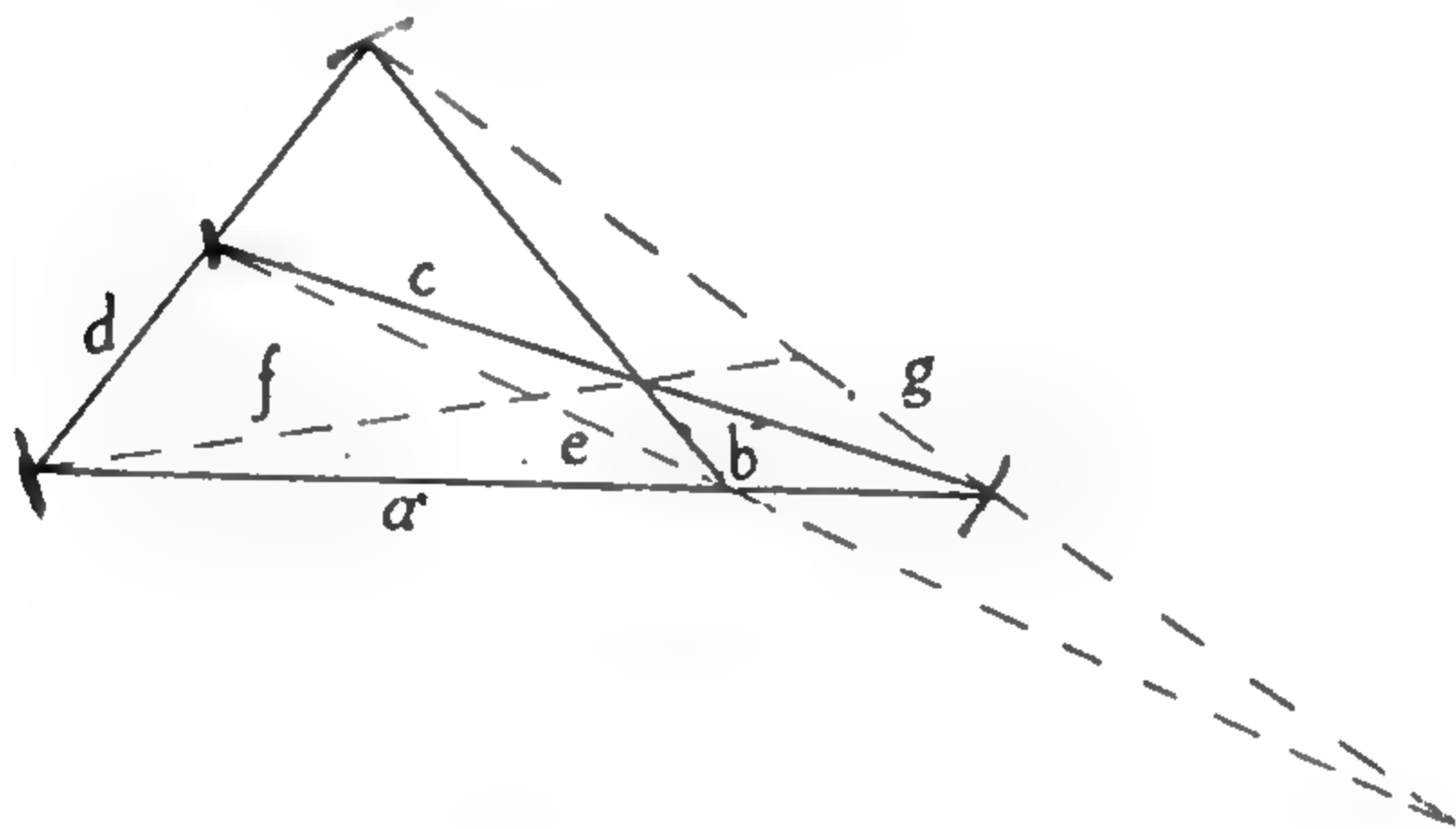


Fig. 7.

The four lines intersect in pairs in six points called the **vertices** of the quadrilateral.

The figure so obtained is called a **complete quadrilateral**.

Two vertices are said to be opposite, if, when one lies on two of the lines, the other lies on the remaining two. The three pairs of opposite vertices in (fig. 7) are $(ab); (cd); (ac); (bd); (ad); (bc)$.

The lines $e.f$, g joining the opposite vertices are called the **diagonal lines**. The triangle formed by the lines $e.f, g$ is called the diagonal triangle of the quadrilateral.

It will be seen that on every side of the quadrilateral there lie three vertices.

Note. There is a remarkable kinship in the wordings of the two descriptions of § 1·83 and 1·84. By a mere interchange of the terms 'point' and 'line', 'collinear' and 'concurrent' 'range' and 'pencil', 'line joining two points' and 'point of intersection of two lines' etc. we can pass from one description to the other. A similar transformation in the wording of a proposition true for any of the two figures, gives a result true for the other. This is a simple example of a very remarkable principle of Modern Geometry, known as the **Principle of Duality**. This principle applies to reciprocal figures such as the complete quadrangle and the complete quadrilateral.

1·9. Principle of Continuity. Modern Pure Geometry is indebted to Algebraical Geometry for one of its most fundamental and useful principles, called the *Principle of Continuity*. This principle introduces imaginary elements in the discussion of pure geometry and asserts that "Theorems concerning real points, and lines may be extended to imaginary points or lines". Thus, if we have to prove a proposition in Pure Geometry concerning a straight line and circle, we prove it when the line intersects the circle in real points and extend the proposition to the case of imaginary intersections by the principle of continuity.

Note 1. It should be clearly borne in mind that this principle of continuity has its foundations in Analytical Geometry. It is, however, not intended to enter into the discussion necessary for the justification of this principle here.

CHAPTER II

PROJECTION AND PERSPECTIVE

✓21. **Projection of a point.** Let P be a point in a plane α . Let π be any other plane and V a point external to both α and π . If VP meets π in P' , P' is said to be the projection of P on π .

The process is called **conical** or **central projection**, V , is called the **vertex** of projection; π , the **plane of projection**, and the section of the planes α and π is called the **axis of projection**.

To avoid confusion, the projection of an element will always be denoted by attaching a dash or suffix to the symbol denoting the given element.

Projection of a plane figure. If P traces out a figure F in a plane α , then P' will trace out another figure F' in π . F' is said to be the projection of F in the plane π from the vertex V .

21. **Projective figures.** If F_1 is the projection of F in the plane π with a vertex V_1 , F_2 is the projection of F_1 in the plane π_2 with a vertex V_2 , and the process is repeated a finite number of times to obtain F_n , the figures F and F_n are said to be **projective**. Evidently the figures $F, F_1, F_2, \dots, F_n$ are projective with one another.

✓**Def.** *Two figures are said to be projective when each can be obtained from the other by a finite number of projections.*

It may be noted that F_n is not the projection of F in the plane π_n . Any figure F_r of the series, however, is the projection of F_{r-1} or F_{r+1} .

✓2. Projection of a straight line. Clearly, a straight line l in α projects into a straight line l' in π , l' being the section of the planes π and Vl .

The lines l and l' which are co-planar must intersect and their point of intersection, being common to the two planes α and π , must lie on their line of intersection, the axis of projection. Consequently

a straight line and its projection intersect on the axis of projection.

✓2.21. The following are immediate corollaries from the definition of projection.

1. There is a one-to-one correspondence between the elements of a figure F and those of its projection F' .

2. If a point P lies on the line l , the projection P' of P lies on the projection l' of l .

3. The line joining two points A, B projects into the line joining A', B' , the projections of A and B .

4. Two intersecting lines PA, PB project into two intersecting lines $P'A', P'B'$, P' being the projection of P .

5. A range of points projects into a range, its base being the projection of the base of the former.

6. A pencil of lines projects into a pencil of lines, its vertex being the projection of the vertex of the former.

✓2.3. Vanishing line. If VP is parallel to the plane of projection π , the projection of P is a point at infinity in π and P is called a vanishing point of the plane α .

The locus of P in the plane α so that VP is parallel to π , is evidently a straight line l , viz., the line of section of α and a plane through V parallel to π . The line l ,

which projects into the line at infinity in π is called the **vanishing line** of the plane α corresponding to the plane π and the vertex V . It is the intersection of the plane α with the plane through the vertex V drawn parallel to the plane of projection π ; the vanishing line l is evidently parallel to the axis of projection.

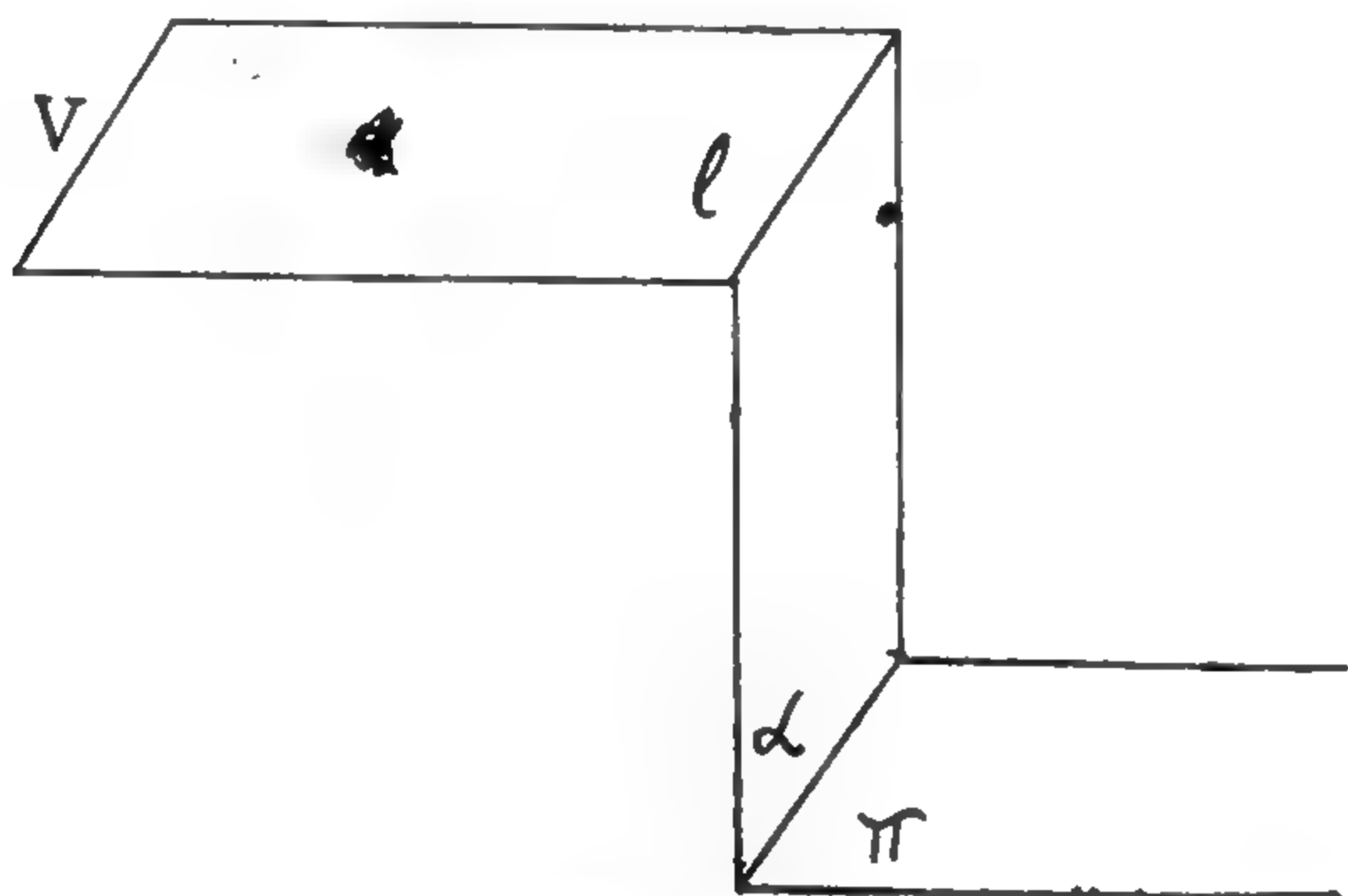


Fig. 11.

The following results follow immediately.

(i) A system of lines in the plane α which concur at a vanishing point of the plane α projects into a system of parallel lines. (for, the point of concurrence of the given system of lines projects into a point at infinity).

(ii) A system of parallel lines in the plane α projects into a system of concurrent lines in π ; the point of concurrence being the projection of the point at infinity in α common to the system of parallel lines.

Note. It will be seen later on that the notion of vanishing line plays an important part in the theory of projection. The student is therefore advised to make himself thoroughly familiar with it.

2.4. Theorem. An angle in a plane α projects on to the plane of projection π into an angle of magnitude equal to that subtended at the vertex of projection V by the portion of the vanishing line intercepted by the arms of the given angle.

Let $\angle BAC$ be any given angle in a plane α and let its arms AB, AC meet the vanishing line l in B and C . We

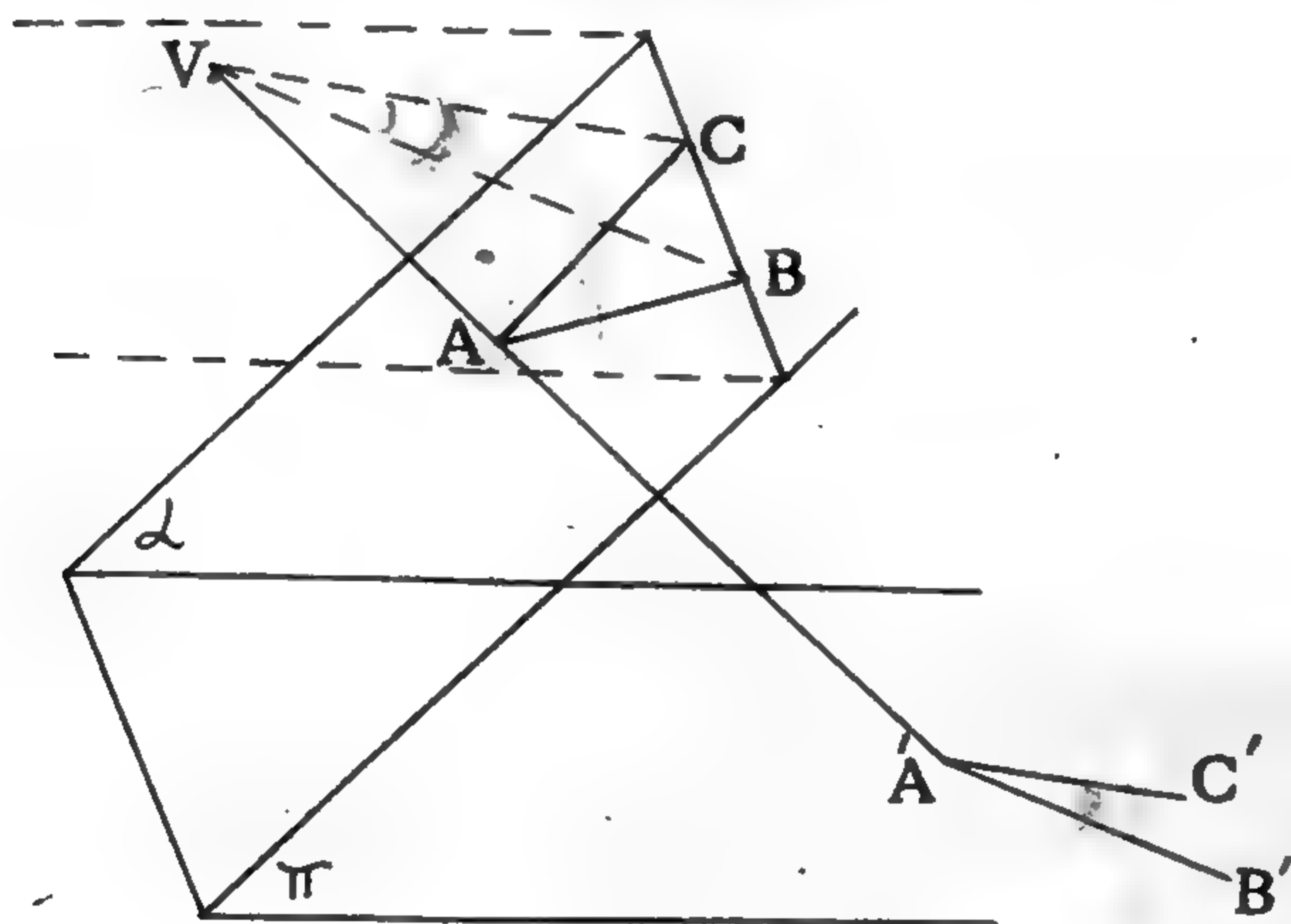


Fig. 12.

have to show that $\angle BAC$ projects into an angle of magnitude $\angle BVC$.

Let the planes VAB and VAC meet the plane of projection π in the lines $A'B', A'C'$ which are, therefore, the projections of the lines AB and AC .

Since the plane VBC is parallel to the plane π , their lines of intersection with the plane VAB are parallel so that

$$VB \parallel A'B'$$

$$\text{Similarly, } VC \parallel A'C'$$

$$\therefore \angle BVC = \angle B'A'C'.$$

2.5. Theorem. *By a proper choice of the vertex of projection V , and the plane of projection Π ,*

(i) any given line l on a given plane α can be projected to infinity, and

(ii) any two given angles BAC , EDF in the plane α can be projected into angles of given magnitude θ and ϕ .

Let the arms of the given angles meet the given line in the points B , C and E , F respectively.

Draw any plane β through the line l . On the segments BC , and EF in the plane β , draw segments of circles containing angles equal to θ and ϕ respectively. Let V be one of their points of intersection.

Taking V as the vertex of projection and any plane parallel to the plane β as the plane of projection, the line l (which is the vanishing line of α) is projected to infinity. The angles BAC and EDF will now project into angles of magnitude θ and ϕ , these being the angles

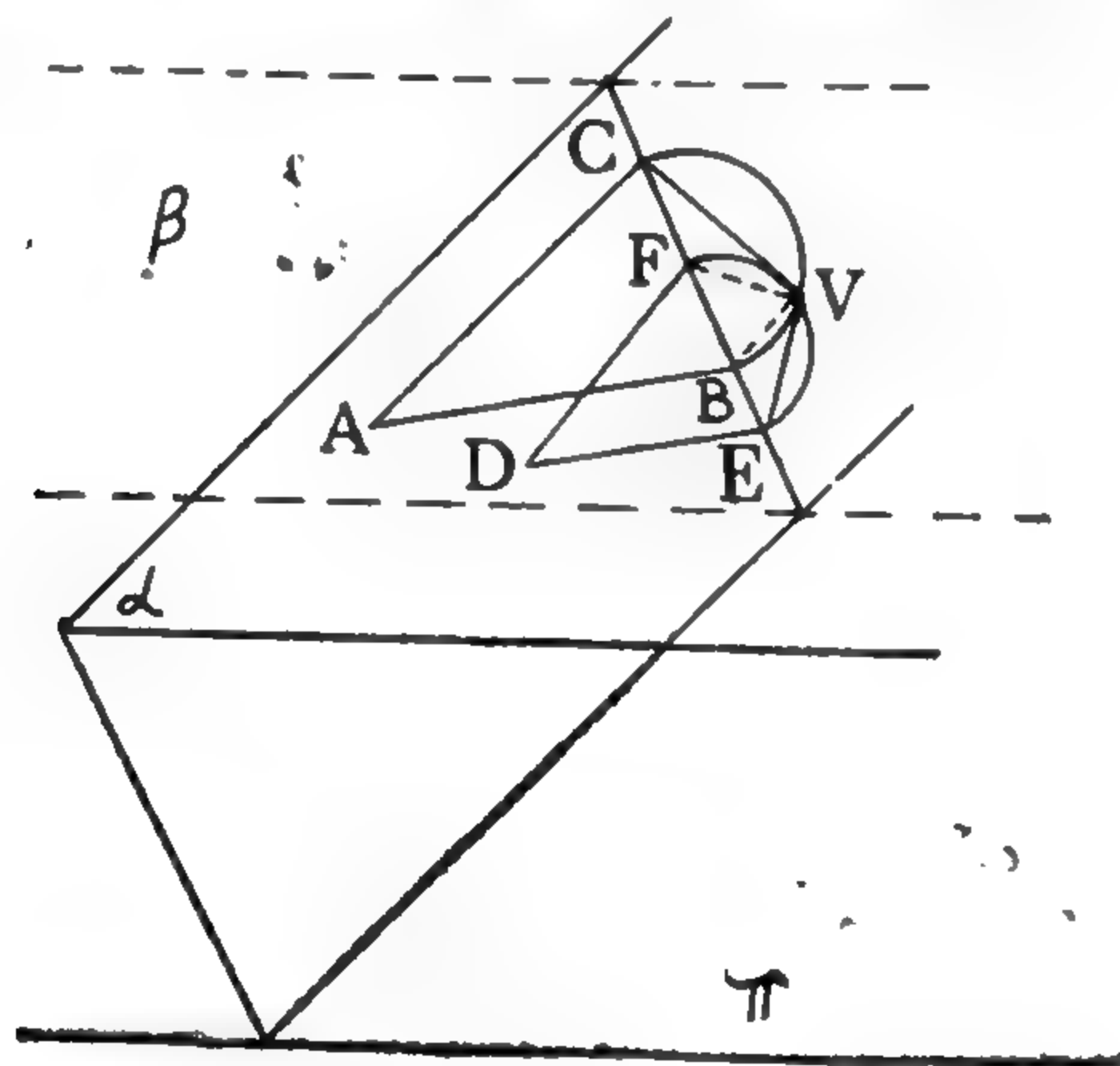


Fig. 13.

which the intercepts BC , EF of their arms on the vanishing line subtend at the vertex of projection V .

Note. The vertex of projection V is obtained as the intersection of two circles which may not intersect at a real point so that in this case V is imaginary and hence also the projection is imaginary. But by the *Principle of continuity* the result still holds good.

There will be two real positions of the vertex of projection in case the two circles intersect in two real points.

Ex. 1. Show that a triangle can be so projected that three concurrent lines drawn through its vertices project into the altitudes.

Let ABC be the given triangle and let AD, BE, CF be three concurrent lines through its vertices.

Choose the vertex of projection and the plane of projection so that the angles BEC and BFC project into right angles. Denoting the

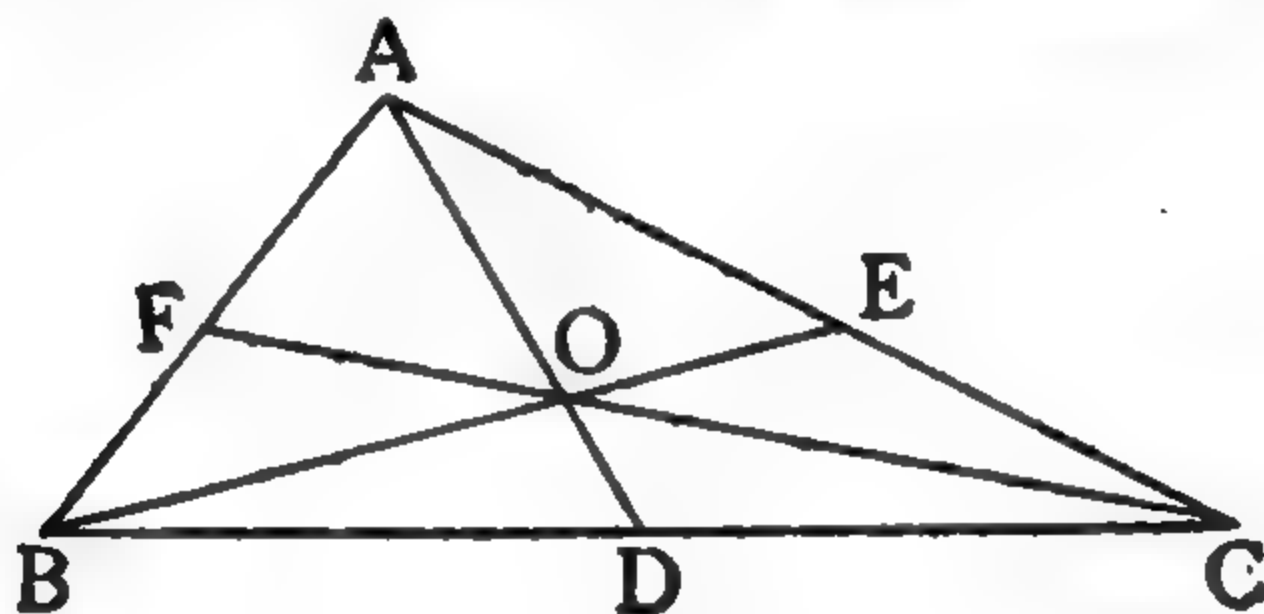


Fig. 14.

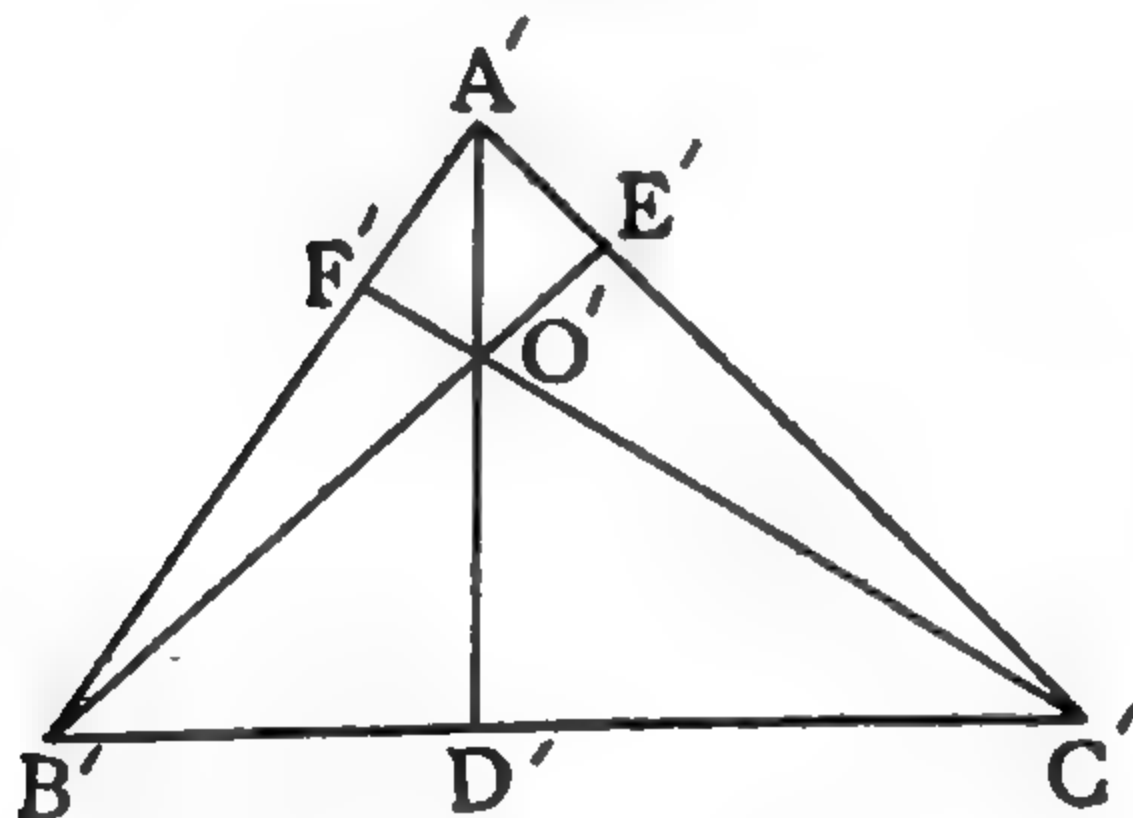


Fig. 15.

projections by corresponding dashed symbols, we see that $B'E'C'$ and $B'F'C'$ are right angles so that $B'E'$ and $C'F'$ are the two altitudes of the triangle $A'B'C'$ which meet at O' , the projection of O . The line AOD projects into $A'O'D'$ and since the altitudes of a triangle are concurrent, $A'D'$ is also an altitude.

Ex. 2. Show that a quadrilateral can be projected into a square.

Let AB, BC, CD, DA be the four sides of the given quadrilateral.

Let AB, CD meet in E , AD, BC in F and AC, BD in G . (See Fig. 6, P. 10.)

Choose the vertex of projection and the plane of projection so that EF is projected to infinity and the angles BAD and AGB are projected

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PROJECTION

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into right angles. Denoting the projections by corresponding dashed symbols, we see that since E', F' are points at infinity,

$$A'B' \parallel D'C' \text{ and } A'D' \parallel B'C'$$

so that $A'B'C'D'$ is a parallelogram.

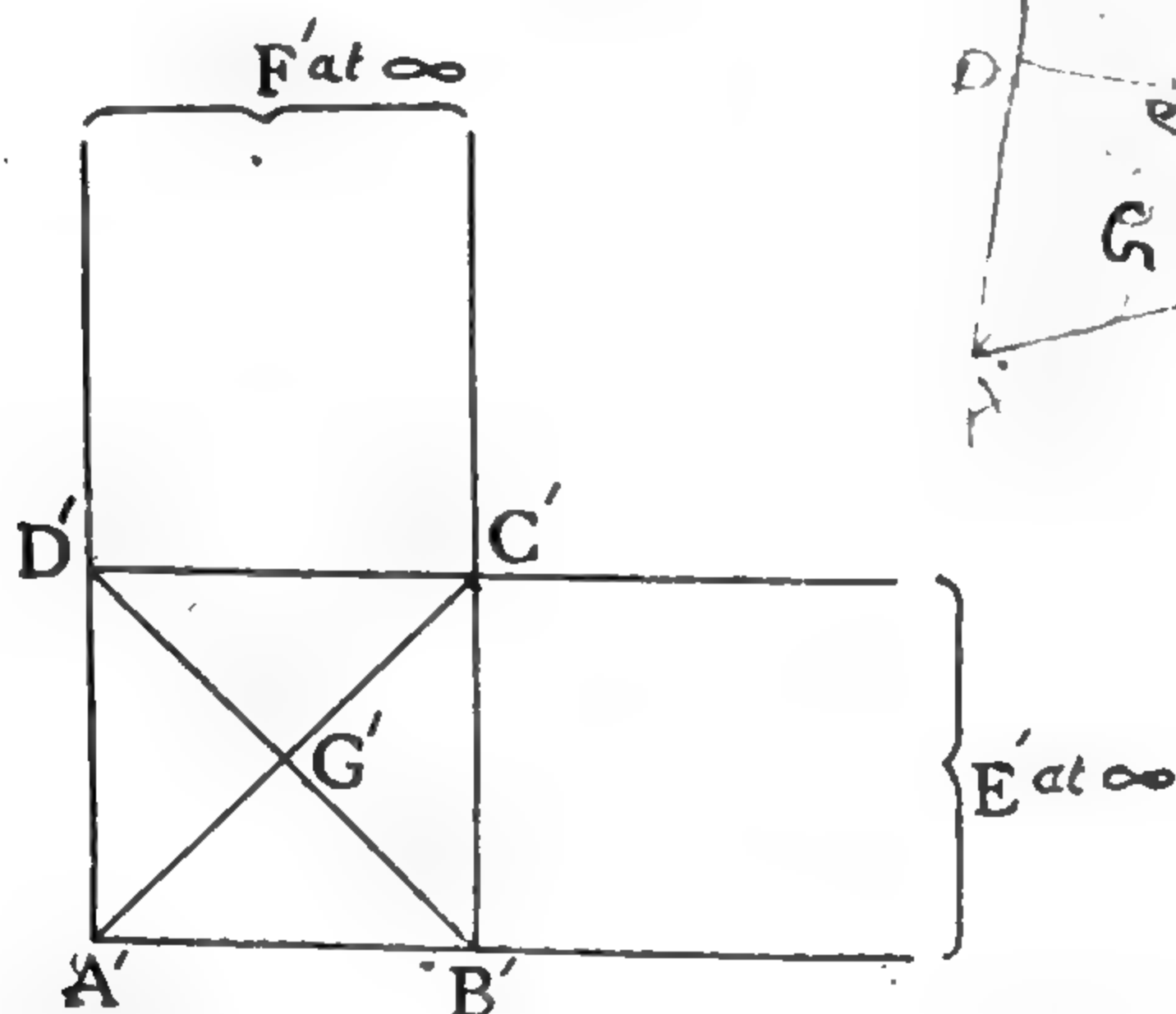


Fig. 16

Again, since $B'A'D'$ is a right angle, the parallelogram must be a rectangle.

Finally, since $\angle A'G'B'$ is a right angle, the rectangle must be a square.

Ex. 3. Show that any triangle can be projected in an equilateral triangle while a line in its plane can be projected to infinity.

Theorem. A range of three points is projective with any other range of three points.

Let A, B, C be a range of three points and let A', B', C' be another range not necessarily coplanar with the first. Join AA' , and take any point V on it.

The lines VAA', VB and VC lie in a plane, say, α . Through A' draw any line lying in the plane α , and let VB, VC meet it in Q and R respectively. The lines $A'B'C'$ and $A'QR$ are co-planar so that QB', RC' which are in their plane, will meet in a point, say,

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V' . Join $V'A'$. Now, the range A, B, C is projective

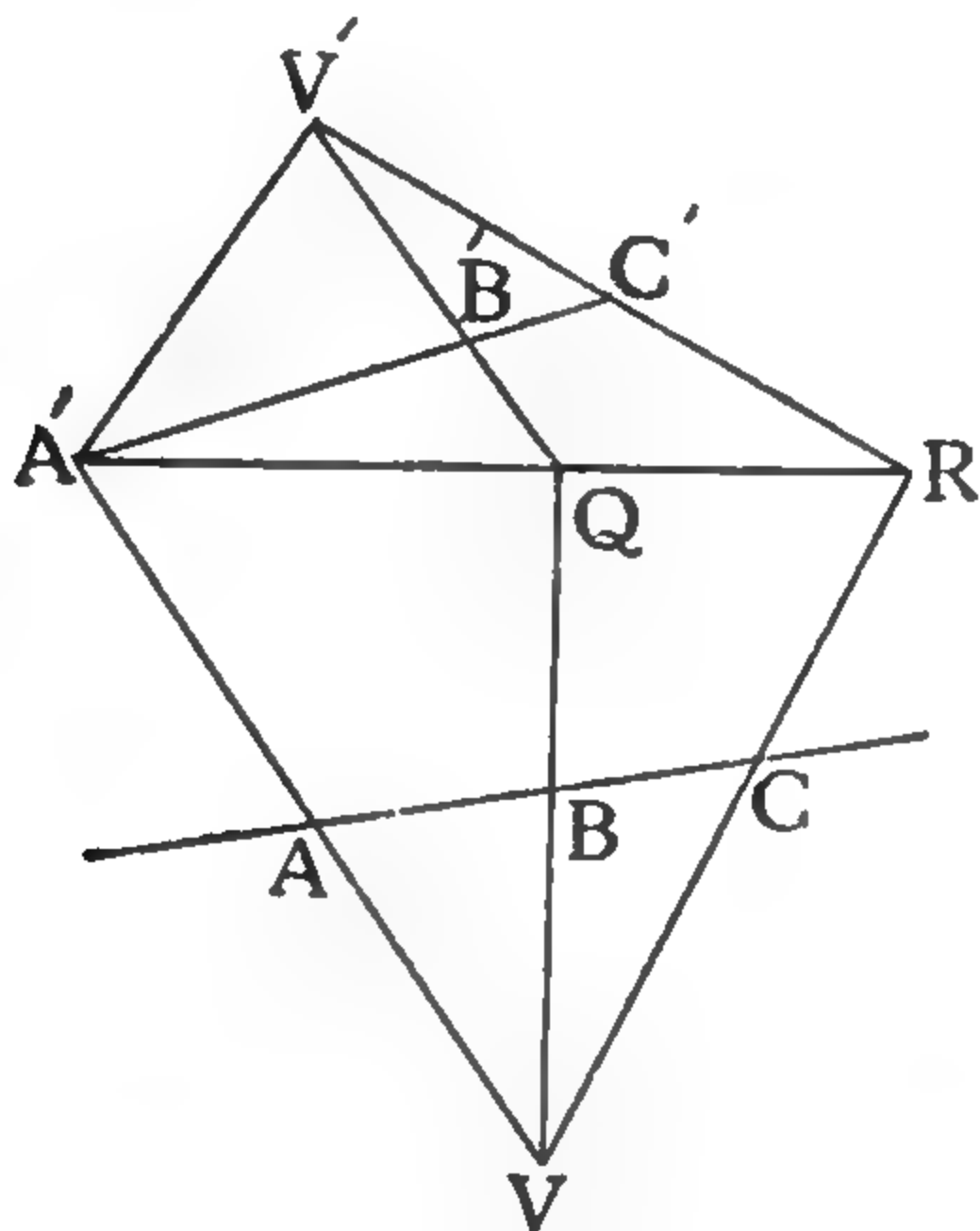


Fig. 17.

with A', Q, R , (with vertex V) which in turn is projective with A', B', C' (with vertex V'). Hence the range A, B, C is projective with the range A', B', C' .

Note. The student should not be led to assume that because a range of three points is projective with any other range of three points in space, a range of four points or for that matter a range of n points is projective with any other range of n points. There is a condition attached to the projectivity of two ranges each consisting of more than three points, but we reserve the discussion of this topic for a later chapter.

2.7. Figures in perspective. A figure consisting of points $A, B, C, \dots, K$ is said to be *in perspective* with another figure consisting of points $A_1, B_1, C_1, \dots, K_1$, if the lines $AA_1, BB_1, CC_1, \dots, KK_1$ are concurrent.

$A, A_1; B, B_1; \dots, K, K_1$ are called *corresponding points* and $AB, A_1B_1; BC, B_1C_1; \dots$ are called *corresponding sides*. The point of concurrence is called the **centre of perspective**. Evidently a figure F in a plane α is in perspective with its projection F' in a plane π , the vertex of projection being the centre of perspective. It does not follow, however, that if two figures are in perspective, each is the projection of the other,

for, two figures can be in perspective even when they are co-planar. We cannot, obviously, speak of one figure as the projection of the other in this case.

✓271. Desargues' theorem on triangles in perspective.

Theorem. *If the vertices of two triangles are in perspective, their corresponding sides intersect in three collinear points, and conversely.*

(1) *Let the two triangles be in the same plane:*

Let ABC , $A_1B_1C_1$ be the two given triangles in

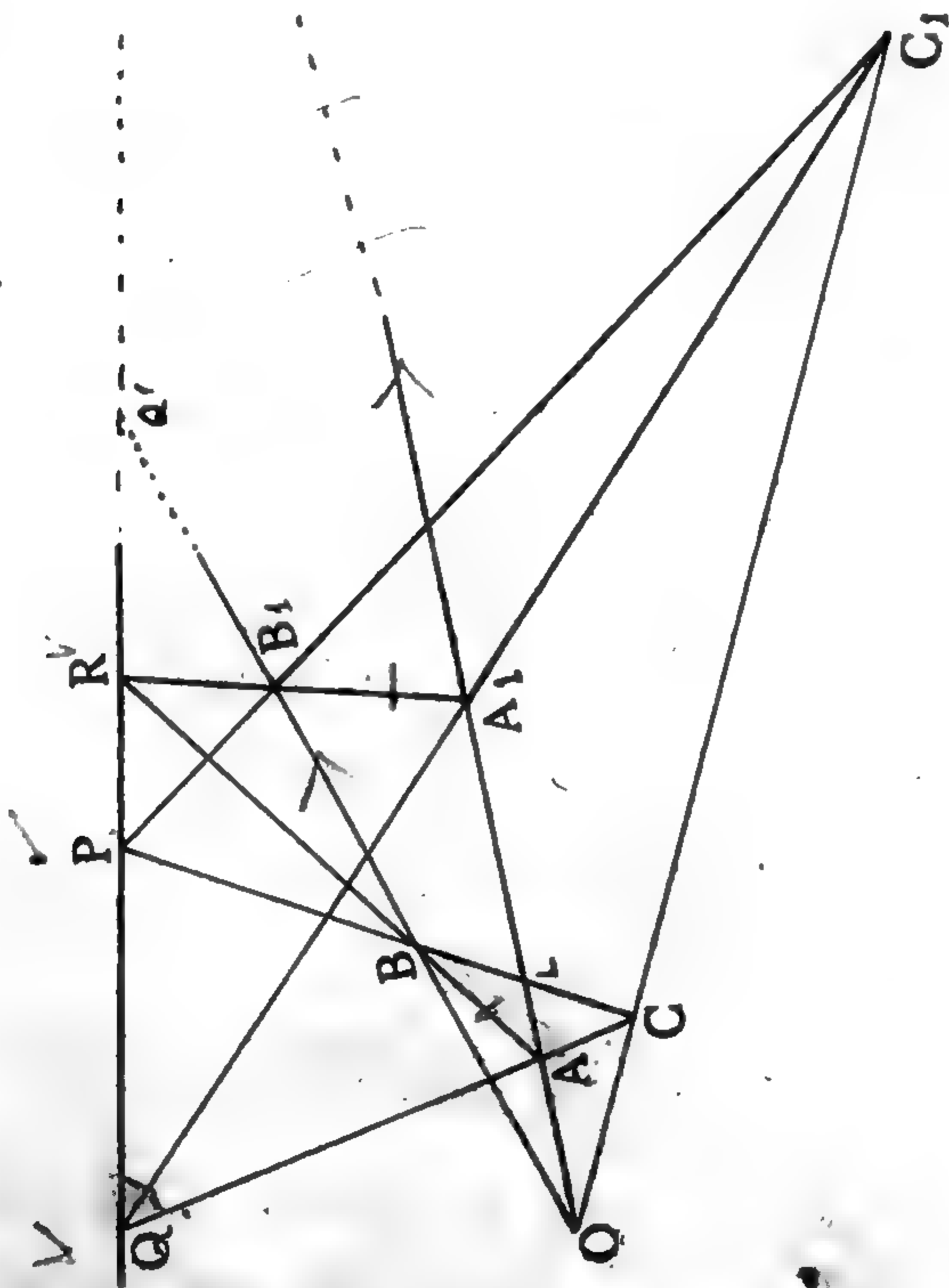


Fig. 18.

perspective so that AA_1 , BB_1 , CC_1 are concurrent in O .

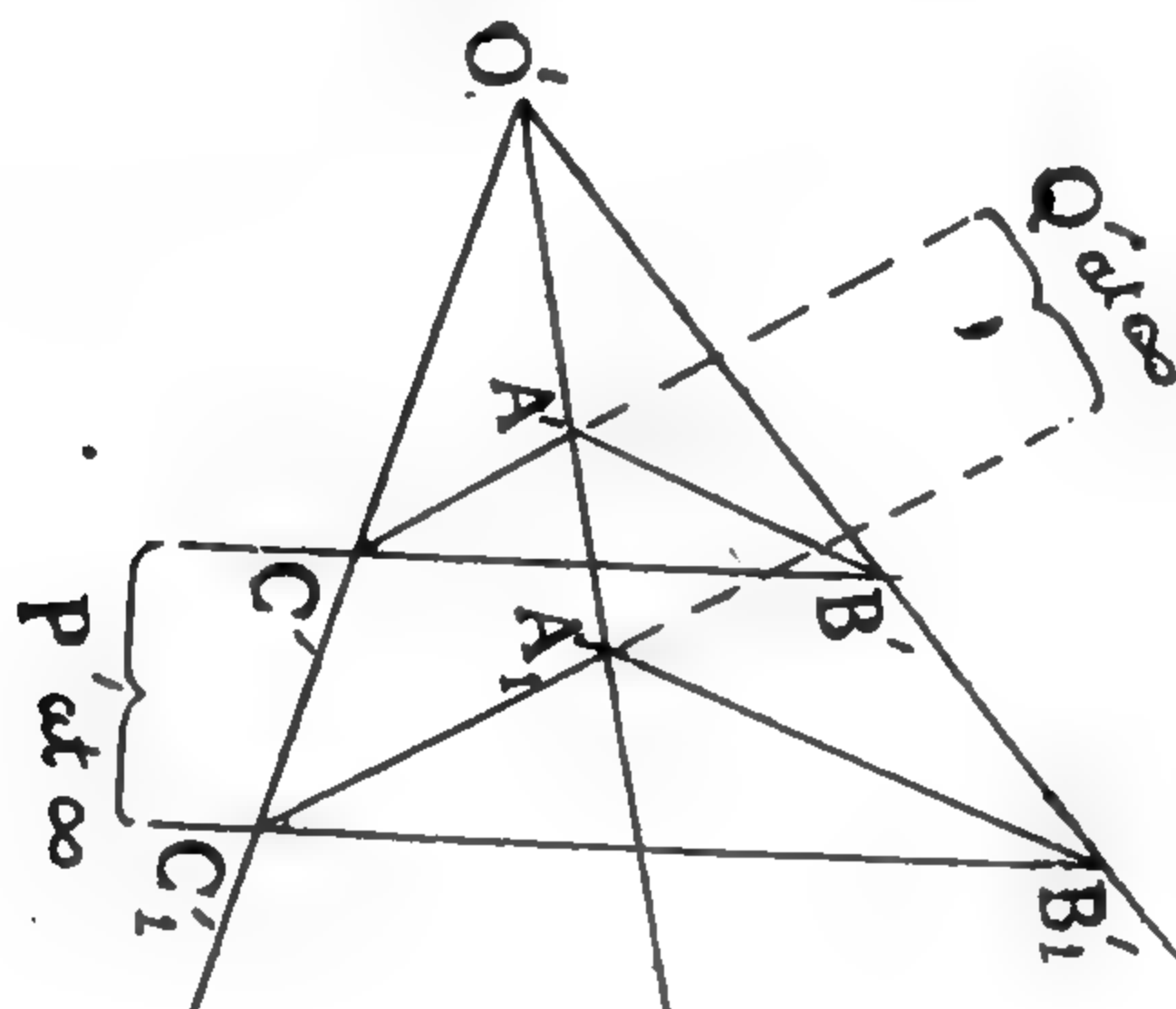


Fig. 19.

Let P , Q , R be the points of intersection of corresponding sides BC , B_1C_1 ; CA , C_1A_1 ; AB , A_1B_1 respectively. We have to show that P , Q , R are collinear. Project the figure so that PQ is projected to infinity and denote the projections by corresponding dashed symbols.

$\therefore P', Q'$ are at infinity,

$\therefore B'C' \parallel B_1'C_1$, and $C'A' \parallel C_1'A_1$

$$\therefore \frac{O'B'}{O'B_1'} = \frac{O'C'}{O'C_1'} \text{ and } \frac{O'C'}{O'C_1'} = \frac{O'A'}{O'A_1'}$$

From these, we have

$$\frac{O'B'}{O'B_1'} = \frac{O'A'}{O'A_1'}$$

$\therefore A'B' \parallel A_1'B_1'$

i.e., R' is at infinity.

Thus because P' , Q' , R' are collinear, lying as they do on the line at infinity, the points P , Q , R must also be collinear.

Conversely, let P , Q , R be collinear. Then AA_1 , BB_1 , CC_1 are concurrent.

Let AA_1, BB_1 meet in O .

Join OC . If it does not pass through C_1 , let it meet A_1C_1 in C_2 .

Let P_1 be the point of intersection of BC and B_1C_2 .

Now the triangles $ABC, A_1B_1C_2$ are in perspective.

Therefore their corresponding sides intersect in collinear points. Thus P_1, Q, R are collinear; i.e., BC meets QR in two different points P and P_1 which is impossible. Therefore P and P_1 coincide.

Therefore B_1C_2 coincides with B_1C_1 i.e., C_2 coincides with C_1 .

Hence AA_1, BB_1, CC_1 are concurrent.

Thus the proposition has been completely proved for triangles in the same plane.

(ii) *Let the triangles be in different planes.*

The lines BC, B_1C_1 lie in a plane, the plane of the two intersecting lines OBB_1, OCC_1 . Hence they must meet. Let P be their point of intersection.

Similarly, CA will meet C_1A_1 and AB will meet A_1B_1 . Let Q and R be their points of intersection respectively. The points P, Q, R which are in the planes of both the triangles must lie on their line of intersection so that they are collinear.

Conversely, Let the points of intersection P, Q, R of the pairs of corresponding sides be collinear. Since BC, B_1C_1 intersect (in P), they are co-planar. Similarly, the pairs of lines CA, C_1A_1 ; AB, A_1B_1 are co-planar.

Thus AA_1, BB_1, CC_1 are the lines of intersection of the three planes $CC_1AA_1, AA_1BB_1, BB_1CC_1$ taken in pairs and since three planes meet in a point, the three lines AA_1, BB_1, CC_1 are concurrent.

Note. Desargue's theorem may also be stated as follows :—

If the lines joining the corresponding vertices of two triangles be concurrent, then the points of intersection of the corresponding sides are collinear.

2.72. The line PQR containing the points of intersection of the corresponding sides of two triangles in perspective, is called the *axis of perspective*.

Triangles whose vertices are in perspective are sometimes said to be *co-polar* : the centre of perspective is then called the *pole* and the axis of perspective is called the *axis*.

Desargue's Theorem can, therefore, be worded thus :

Two co-polar triangles are co-axal and conversely.

Note 1. Triangles in *plane* perspective are sometimes called *homologous* triangles. The centre of perspective in that case is called the *centre of homology* and the axis of perspective is called the *axis of homology*.

Each triangle is said to be a *homologue* of the other.

Note. 2. If two triangles in different planes be in perspective, then each can be thought of as the projection of the other with centre of perspective as the vertex of projection, and the axis of perspective as the axis-of projection.

Ex. 1. Three concurrent lines AD, BE, CF are drawn through the vertices A, B, C of a triangle ABC to meet the opposite sides in D, E, F ; the three lines EF, FD, DE meet the sides BC, CA, AB respectively in P, Q, R . Show that P, Q, R are collinear.

Ex. 2. *When three triangles are two by two in perspective, and have the same axis of perspective, then their three centres of perspective are collinear.*

Sol. Let $A_1B_1C_1, A_2B_2C_2$ and $A_3B_3C_3$ be three triangles, and let the line l be their common axis of perspective. Let O_1 be the centre of perspective of the $\triangle$ s $A_2B_2C_2, A_3B_3C_3$; O_2 that of $A_3B_3C_3, A_1B_1C_1$ and O_3 that of $A_1B_1C_1, A_2B_2C_2$.

By hypothesis, the point of intersection of the lines A_1B_1 and A_2B_2 lies on l and the point of intersection of A_2B_2 and A_3B_3 also lies on l . Hence the lines A_1B_1, A_2B_2, A_3B_3 are concurrent.

Therefore the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are co-polar and hence are also co-axal. Thus the points of intersection O_1, O_2, O_3 of the pairs of corresponding sides A_2A_3, B_2B_3 ; A_3A_1, B_3B_1 ; A_1A_2, B_1B_2 , of the triangles $A_1A_2A_3, B_1B_2B_3$ are collinear.

2.8. Projective properties. *The principle of projection* constitutes an important instrument for studying such properties of figures as remain unaltered in projection. Such properties are known as **Projective properties***. For example, the property of collinearity of a certain system of points is a projective one. Again, the property of a line intersecting a curve in two points is a projective property. On the other hand, if A, B, C be three collinear points and A', B', C' be their projections, then, in general,

$$\frac{AB}{BC} \neq \frac{A'B'}{B'C'}$$

so that the ratio of two collinear segments is not projective.

Similarly, the property of the circle that a chord is perpendicular to its bisector through the centre, is not a projective property.

In general, properties of figures concerning magnitudes of angles, lengths, areas etc—known as metrical properties—alter on projection, but descriptive properties are all projective.

It will, however, be seen in Chapter IV that there exists an important *metrical* concept (known as *Cross Ratio*) which is projective. Consequently metrical properties of figures which are expressible in terms of cross ratios can be studied through projection.

Projective Geometry includes within its scope all those descriptive and metrical properties of geometrical figures which remain unaltered on projection.

* Desargue's theorem for plane triangles, embodying, as it does, the projective relations of concurrency and collinearity, has been proved here with the help of projection. Another proof, not employing the notion of projection is given in Chapter III.

Exercises.

1. Two angles whose arms meet the vanishing line in the same points are projected into angles which are equal to one another.
2. Show that three given angles can be projected into angles of the same magnitude α .
3. Show that if a triangle PQR be projected from two points O, O' , not lying in the same plane, into two triangles $ABC, A'B'C'$ in a plane π not passing through any of the points P, Q, R, O, O' , then these two triangles are in perspective and the pole is the intersection of OO' with π . Where is the axis of perspective?
4. ABC are three fixed collinear points; PQR is a variable triangle such that P, Q lie on fixed lines and QR, RP, PQ pass through A, B, C respectively, prove that the locus of R is a straight line.
5. If the vertices of a triangle move on fixed concurrent straight lines and if two of the sides pass through fixed points, the third side also passes through a fixed point collinear with the other two.
6. Two quadrangles $ABCD$ and $A_1B_1C_1D_1$ are such that the intersection of the five pairs of corresponding sides like (AB, A_1B_1) lie on a straight line; show that the sixth pair also intersect on the same straight line.
7. Two triangles ABC and $A'B'C'$ are in perspective. If $BC', B'C$ meet in A_1 , $CA', C'A$ in B_1 ; $AB', A'B$ in C_1 , then the triangle $A_1B_1C_1$ is in perspective with each of the given triangles and the three triangles have a common axis of perspective.
8. If EFG be the diagonal triangle of a complete quadrangle $ABCD$ and the sides of EFG also meet the sides of the quadrangle in six other points I, J, K, L, M, N ; show that I, J, K, L, M, N are the six vertices of a complete quadrilateral having EFG for its diagonal triangle.
9. When three triangles are two by two in perspective and have the same centre of perspective, then their axes of perspective are concurrent.
10. A, B, C ; A', B', C' are two sets of three collinear points in, the same plane; prove that the meets of $AB', A'B$; $BC', B'C$; $CA', C'A$, are collinear.

EXERCISES

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11. $OABC, OA'B'C'$ are two straight lines such that AA', BB', CC' are parallel ; $AB', A'C$ meet in P ; $A'B, AC'$ meet in Q ; show that PQ is parallel to AA' .

12. Given the planes α, π and the vertex of projection V ; show that there are two points M, N in α such that any angle in α whose apex is at M or N is unaltered in size by projection, also show that these two points subtend a right angle at the vertex of projection.

13. Show how to project from one given plane to another so that the angles at a given point in the first are unaltered by projection.

14. XY is the axis of projection and l the vanishing line of a plane α ; $X'Y'$ is the reflection of XY in l ; prove that the length of any segment in $X'Y'$ is unaltered by projection. (This line is called an equi-segmental line).

Sup 15. Show that two plane triangles in perspective can be projected into equilateral triangles. *1976*

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CHAPTER III

SIGNED GEOMETRICAL MAGNITUDES.

Ceva's and Menelaus' Theorems.

3.1. The concepts of Geometry, which are capable of measurement are known as Geometrical magnitudes. *Length*, *Area*, and *Angle* are the three geometrical magnitudes with which we shall be concerned in this book. To measure a geometrical magnitude is equivalent to associating some definite number with it. In elementary classical Geometry, this number is always positive, *i.e.*, we are concerned with absolute magnitudes and have, ordinarily, as a result of this limitation, to distinguish between numerous cases of the same phenomenon according to the *appearance* of the corresponding figure. In modern geometry, however, it is usual to regard a geometrical magnitude as equipped with a sign so that a number associated with it may be positive or negative. This is, of course, done by adopting some suitable conventions, as will be seen below. The student will easily appreciate the great advantage thus gained over classical geometry. As a matter of fact, the use of signs in geometry is but another step towards the achievement of a certain uniformity in our statements—one of the greatest trends of modern mathematics—whatever the *appearance* of the figure be; the various cases of elementary classical Geometry just merging into one.

3.2. **Segments along a line.** The fact that a segment lying along a line can be described in two different senses enables us to associate a sign with it so that a segment is positive or negative according as it is described in one sense or the

other. Thus if A and



Fig. 20.

A and B are two points on a line, the direction of motion from A to B is opposite to the one from B to A so that we have

$$AB = -BA$$

or

$$AB + BA = 0.$$

The convention as to which of the two directions is to be positive is purely *arbitrary*, but once adopted, it is *binding* throughout the discussion of a problem. If a line be drawn parallel to the printed lines of the page, then the convention usually adopted is that a segment measured from left to right is positive and that measured from right to left is negative. It should, however, be observed that the various formulæ about segments are independent of the particular convention adopted and retain their validity even if the opposite convention be adopted. The only essential point is that the segments BA and AB have opposite signs. It does not matter whether AB or BA is taken as positive in a particular problem.

Note. Two segments lying along two *different* lines cannot be compared as to their signs except when the lines are parallel. It is only the segments lying along the same line or along parallel lines which can be compared in relation to signs. Of course, they can always be compared as regards their absolute magnitudes.

One very *useful* consequence of associating a sign with a segment is the following :

If O, A, B be three collinear points, then

$$AB = AO + OB$$

in whatever order the points may lie along the line.

The segment AB can be conceived to have been *stepped* in two instalments. (i) from A to O and (ii) from O to B , a step being positive or negative according as it is taken in the one or the other direction. Thus with

this new meaning attached to the segments and their signs, we have

$$AB = AO + OB.$$

and since $AO = -OA$,

this can be written as

$$\begin{aligned} AB &= -OA + OB \\ &= OB - OA. \end{aligned}$$

The same result will also be apparent if we examine the various possibilities in which the three points may lie on a line in relation to each other. As it is, there are in all six possibilities, (depicted below), three points giving rise to $3! = 6$ arrangements.

Fig. 21



Fig. 23



Fig. 22.



Fig. 24.

Fig. 25



Fig. 26.

For example, in the fig. 26,

$$AB = AO - BO = AO + OB = OB - OA.$$

Inserting the origin. By means of the result established above, we can express any number of lengths lying along a line in terms of lengths measured from some fixed point of the line. This process is known as *Inserting the origin* at the fixed point. Thus, by inserting the origin at O , we obtain

$$AB = OB - OA.$$

✓ **Ex. 1.** If M be the middle point of AB and O be any other point on the line, then

$$OM = \frac{1}{2}(OA + OB).$$



Fig. 27



Fig. 28.

We are given that $AM = MB$.

Inserting the origin at O , we have

✓ **Ex. 2.** $OM - OA = OB - OM$ whence $OM = \frac{1}{2}(OA + OB)$.

A point P on the line AB is such that

$$\frac{AP}{PB} = \frac{m}{n},$$

If O is any other point on the line, show that

$$OP = \frac{m \cdot OB + n \cdot OA}{m + n} \quad \left\{ \begin{array}{l} \text{If } O \text{ is the origin, then } OA = x, OB = y, \text{ and } OP = z. \\ \text{Then } z = \frac{m \cdot y + n \cdot x}{m + n} \end{array} \right.$$

Ex. 3. D, D' are two points on a line AB such that

$$\frac{AD}{DB} = \frac{AD'}{D'B'}$$

show that they must coincide.

Euler's Theorem.

✓ **Ex. 3.** If A, B, C, D be four collinear points, then

$$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0$$

in whatever order the points may lie on the line.

Inserting the origin at A , we see that

$$\begin{aligned} & AB \cdot CD + AC \cdot DB + AD \cdot BC \\ &= AB(AD - AC) + AC(AB - AD) + AD(AC - AB) = 0. \end{aligned}$$

Note. The importance of the Euler's theorem lies in the fact that it gives a relation between ${}^4C_2 = 6$ segments determined by four collinear points.

✓ **Ex.** If A, B, C be any three collinear points and AL, BM, CN , their distances perpendicular, or in any common direction, from any given line p , then

$$BC \cdot AL + CA \cdot BM + AB \cdot CN = 0.$$

Exercises.

1. If AB and $A'B'$ be any two segments on the same line, and C, C' , their two middle points, show that

$$CC' = \frac{1}{2}(AA' + BB') = \frac{1}{2}(AB' + BA')$$

2. A segment AB is divided at a point G , so that

$$m \cdot AG + n \cdot BG = 0,$$

and O is any other point on the line AB , then prove that

$$m \cdot OA + n \cdot OB = (m + n)OG,$$

$$m \cdot OA^2 + n \cdot OB^2 = m \cdot AG^2 + n \cdot BG^2 + (m + n)OG^2.$$

3. The base BC of a triangle ABC is divided at G , so that

$$m \cdot BG + n \cdot CG = 0$$

then prove that

$$m \cdot AB^2 + n \cdot AC^2 = m \cdot BG^2 + n \cdot CG^2 + (m + n)AG^2.$$

Also, generalise this result.

(b) If a, b, c be the lengths of the sides of a triangle, show that the locus of a point P which moves such that

$$a \cdot PA^2 + b \cdot PB^2 + c \cdot PC^2 \text{ is constant}$$

is a circle concentric with the incircle of the triangle.

4. If A, B, C, D etc. be n points on a line, and O , the point on the line for which

$$OA + OB + OC + OD + \text{etc.} = 0.$$

and P any other point on the line, then, whatever be the position of P ,

$$PA + PB + PC + PD + \text{etc.} = n \cdot PO$$

$$PA^2 + PB^2 + PC^2 + PD^2 + \text{etc.} = OA^2 + OB^2 + OC^2 + OD^2 + \text{etc.} + n \cdot OP^2$$

(The point O defined here is known as the mean centre of the points A, B, C, D etc.)

5. If A, B, C, D , etc., and A', B', C', D' etc., be two sets, each consisting of n points, lying on the same line, O and O' their mean centres, then

$$OO' = \frac{1}{n} (AA' + BB' + CC' + DD' + \text{etc.})$$

6. If A, B, C, D etc., and A', B', C', D' etc., be any two sets of points lying on the same line and n, n' their number of points, and O, O' their mean centres, then show that

$$OO' = \frac{\sum (AA')}{nn'}$$

every point of one system being combined in the summation with every point of the other.

✓ If A, B, C be three collinear points, in any order whatsoever and P any other point, then show that

$$BC \cdot AP^2 + CA \cdot BP^2 + AB \cdot CP^2 = -BC \cdot CA \cdot AB.$$

(Stewart's Theorem)

8. If A, B, C be three collinear points, in any order whatsoever, and l_1, l_2, l_3 , the lengths of the three tangents drawn from them to any circle, then

$$\frac{l_1^2}{AB \cdot AC} + \frac{l_2^2}{BC \cdot BA} + \frac{l_3^2}{CA \cdot CB} = 1.$$

3.4. Signs of Areas. Any closed curve which does not intersect itself encloses an area. The fact that the curve (known as *contour*) enclosing an area can be described by a moving point in two different ways depending upon the direction—clockwise or counter-clockwise enable us to consider signed areas so that areas whose boundaries are described in opposite directions have opposite signs.

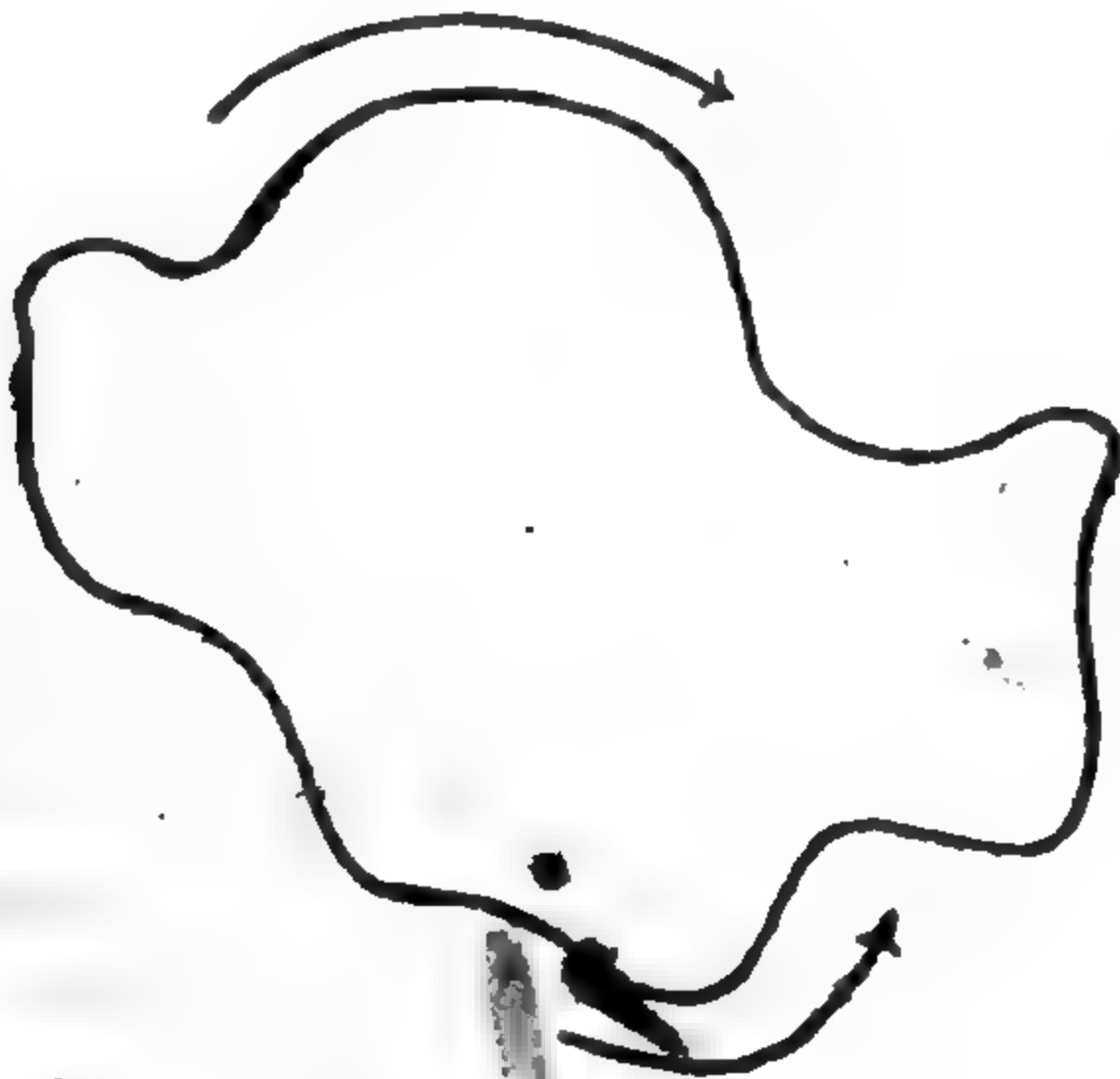


Fig. 29.

The convention usually adopted is that an area is positive or negative according as its boundary is described

by a moving point in the counter-clockwise or clockwise direction.

In this book, we shall be concerned with the areas of triangles only. In this connection, it may be remembered that the order in which the boundary of a triangle is described is indicated by the order in which its vertices are written. Thus we have

$$\Delta ABC = -\Delta BAC.$$

where the symbol ΔABC denotes the area of the triangle ABC .

A cyclic interchange of the order of the letters does not affect the sign of the area. Thus,

$$\Delta ABC = \Delta BCA = \Delta CAB$$

§41. Two important results follow from above.

If B, C, D , be three collinear points, lying in any order whatever, and A is a point not collinear with them, then

$$(i) \quad -\frac{BD}{DC} = \frac{\Delta BDA}{\Delta DCA}.$$

$$(ii) \quad \Delta ABC = \Delta ABD + \Delta ADC,$$

$$\text{or } \Delta ADC = \Delta ABC - \Delta ABD.$$

(i) Clearly the segments BD, DC have the same or opposite signs according as the areas of the triangles ABD and ADC have the same or opposite signs so that the two ratios will have the same sign. The truth of the equality so far as absolute magnitudes are concerned may be seen by drawing the perpendicular from A on BC .

(ii) This is obvious. The second statement of the result may be compared with the method of inserting an origin in a range.

3.5. Signs of angles. The fact that in order to describe an angle, a line starting from a given position may rotate in one or the other direction (clockwise or

counter-clockwise) enables us to consider signed angles. The convention usually adopted is that an angle described is positive or negative according as the line revolves in the counter-clockwise or clockwise direction.

The symbol $\angle AVB$ denotes the angle which VA

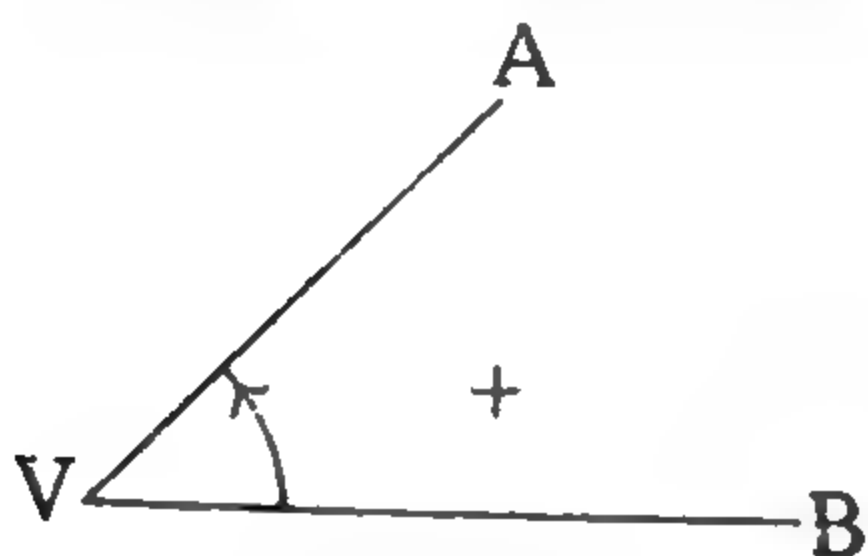


Fig. 30

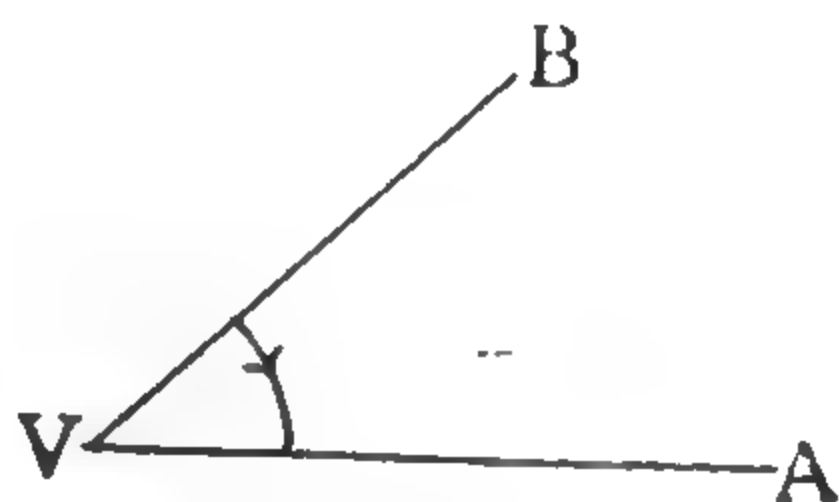


Fig. 31

makes with VB , VA being the final position of the revolving line. If we confine ourselves to angles which are, in absolute value, less than two right angles (we shall be considering only such angles in this book), we may notice that the $\angle AVB$ is positive or negative according as V is on our right or left, when we move from A to B (the final to the initial position) along the line AB , or more simply still, according as the area of the $\triangle AVB$ is positive or negative.

It should be carefully remembered that for angles, which, in absolute magnitude, are less than two right angles, an angle and its sine are of the same sign.

If ABC is any triangle, we have from elementary Trigonometry.

$$\triangle ABC = \frac{1}{2} BA \cdot BC \cdot \sin \angle ABC.$$

so that the equality holds as regards sign also, because $\sin \angle ABC$ has the same sign as $\triangle ABC$.

Thus we see that the conventions of signs for angles and areas do not lead to any contradiction.

3.6. Test for the concurrency of lines drawn through the vertices of a triangle. Ceva's Theorem.

If D, E, F be three points lying on the sides BC, CA, AB respectively of a triangle ABC , then the

necessary and sufficient condition for the three lines AD , BE , CF to be concurrent is

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1.$$

The condition is necessary. Let AD , BE , CF concur in O . Draw YAX parallel to BC , and let it meet BE , and CF in X and Y respectively.

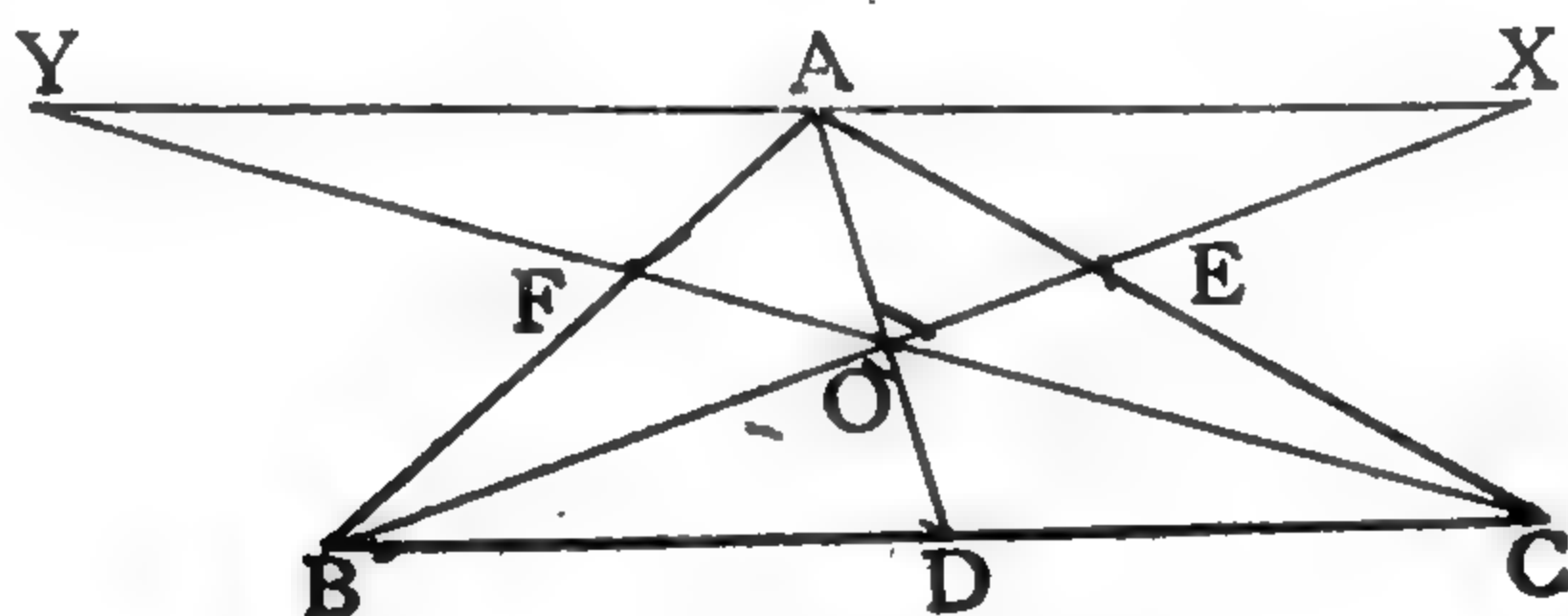


Fig. 32.

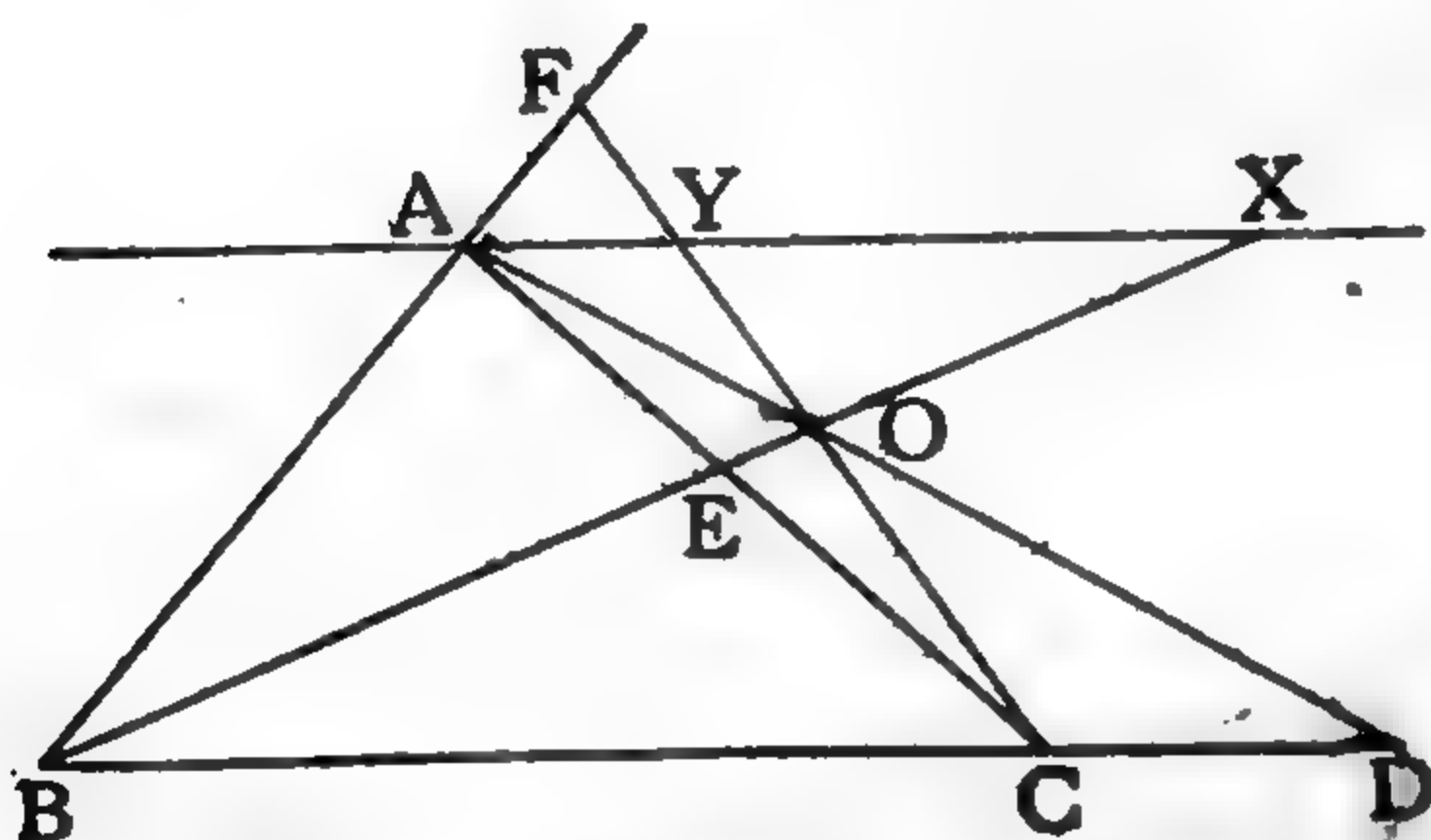


Fig. 33.

From the similar triangles BOD , XOA , with due regard to the signs of the segments, we have

$$\frac{BD}{AX} = \frac{DO}{OA};$$

Similarly, from $\triangle s$ DOC , AOY , we have

$$\frac{CD}{AY} = \frac{DO}{OA}.$$

$$\therefore \frac{BD}{AX} = \frac{CD}{AY} \text{ or } \frac{BD}{CD} = \frac{AX}{AY} \quad \dots (i)$$

From the similar triangles BCE and AEX ,

$$\frac{CE}{AE} = \frac{CB}{AX} \quad \dots \quad \dots (ii)$$

From the similar triangles BCF and FAY

$$\begin{aligned} \frac{AF}{BF} &= \frac{AY}{BC} \\ &= -\frac{AY}{CB} \quad \dots \quad \dots (iii) \end{aligned}$$

From (i), (ii), (iii) by multiplication, we obtain

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1.$$

The condition is sufficient. Let D, E, F be three points on the sides BC, CA, AB respectively, such that

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1. \quad \dots (1)$$

Then AD, BE, CF are concurrent.

Let BE, CF meet in O , and let AO meet BC in D' .

Since AD', BE, CF are concurrent, we have

$$\frac{BD'}{CD'} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1. \quad \dots (2)$$

From (1) and (2),

$$\frac{BD}{CD} = \frac{BD'}{CD'}$$

or

$$BD \cdot CD' = BD' \cdot CD$$

or

$$BD(BD' - BC) = BD'(BD - BC),$$

(inserting the origin at B).

whence $BD = BD'$

so that D and D' coincide.

Hence AD must pass through O

3.61. That the condition is necessary can also be established thus: (Refer. § 3.41, P. 34.)

$$\frac{BD}{CD} = \frac{\triangle BDA}{\triangle CDA} = \frac{\triangle BDO}{\triangle CDO} = \frac{\triangle BDA - \triangle BDO}{\triangle CDA - \triangle CDO} = \frac{\triangle BOA}{\triangle COA}$$

Similarly

$$\frac{CE}{AE} = \frac{\triangle COB}{\triangle AOB} \quad \text{and} \quad \frac{AF}{BF} = \frac{\triangle AOC}{\triangle BOC}$$

$$\begin{aligned} \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} &= \frac{\triangle BOA}{\triangle AOB} \cdot \frac{\triangle COB}{\triangle BOC} \cdot \frac{\triangle AOC}{\triangle COA} \\ &= (-1)(-1)(-1) = -1. \end{aligned}$$

3.7. Test for the collinearity of three points lying on the sides of a triangle. **Menelaus' Theorem.** If D, E, F be any three points lying on the sides BC, CA, AB respectively of a triangle ABC , then the necessary and sufficient condition that these points be collinear is that

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1.$$

The condition is necessary. Let the three points D, E, F be collinear.

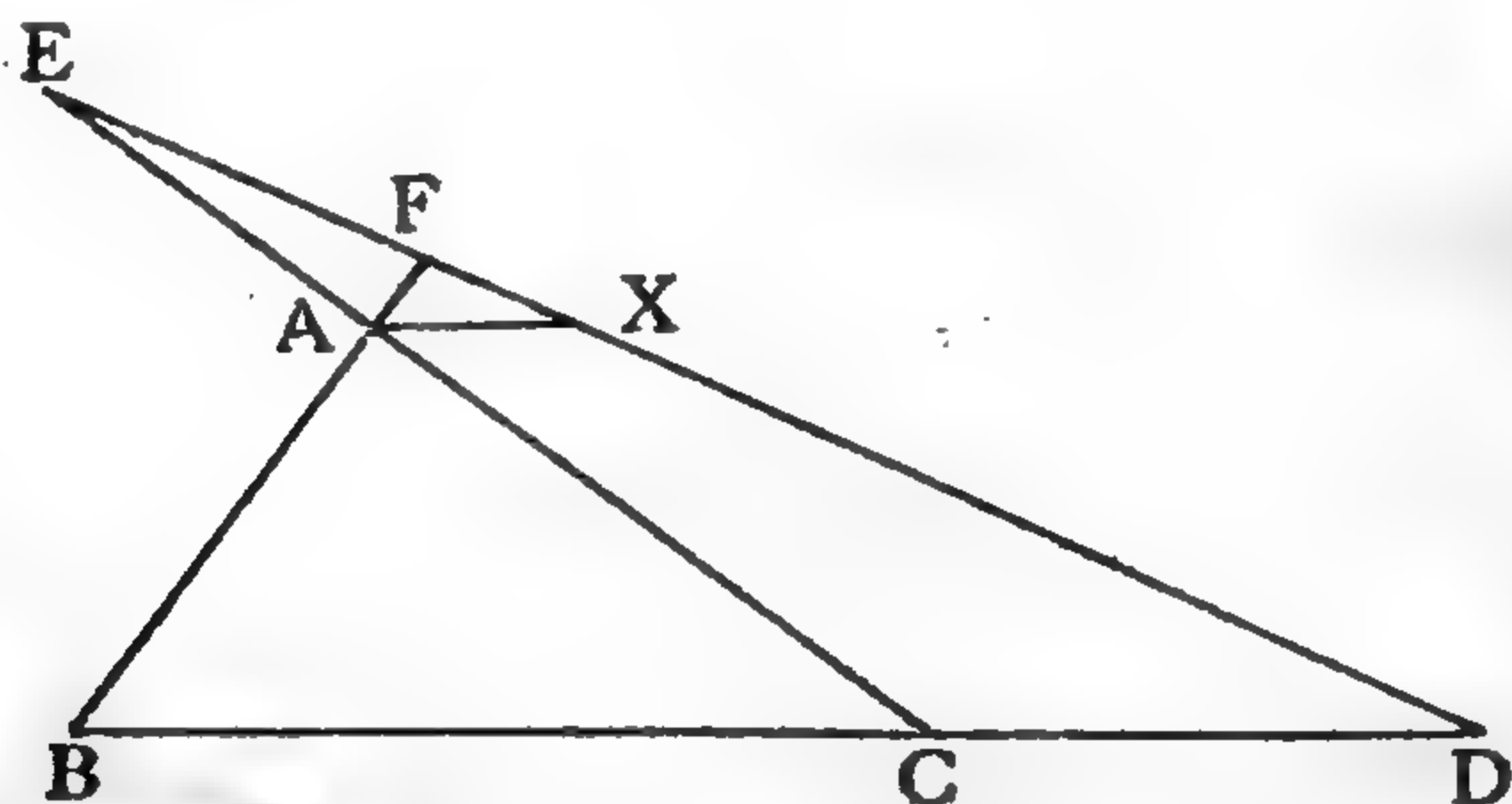


Fig. 34.

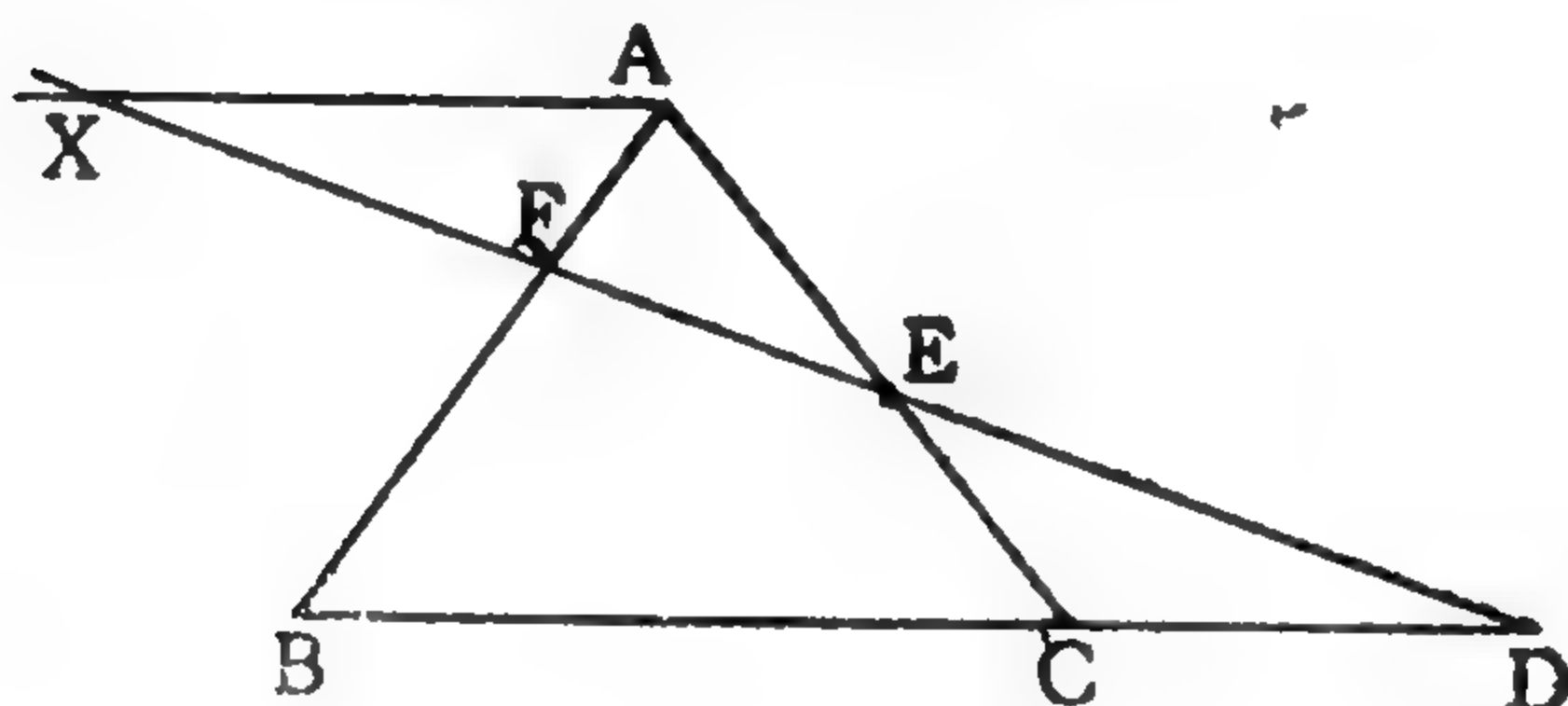


Fig. 35.

Draw AX parallel to BC meeting DEF in X . Then, from the similar $\triangle$ s CED , XAE , (with due regard to sign), we have

$$\frac{CE}{AE} = -\frac{CD}{XA}$$

and from $\triangle$ s BDF , XAF ,

$$\frac{AF}{BF} = -\frac{XA}{BD}$$

$$\therefore \frac{CE}{AE} \cdot \frac{AF}{BF} = \frac{CD}{BD}$$

$$\text{or } \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1.$$

The condition is sufficient. Let D, E, F be three points on the sides BC, CA, AB respectively of a triangle ABC such that

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1. \quad \dots \quad \dots (1)$$

Then the three points are collinear.

Let EF meet BC in D' .

Then, since D', E, F are collinear, therefore

$$\frac{BD'}{CD'} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1. \quad \dots \quad \dots (2)$$

From (1) and (2).

$$\frac{BD}{CD} = \frac{BD'}{CD'}$$

or $BD \cdot CD' = BD' \cdot CD.$

or $BD \cdot (BD' - BC) = BD' \cdot (BD - BC)$

(inserting the origin at B).

whence $BD = BD'.$

so that D and D' coincide.

Hence D, E, F are collinear.

Note. We have seen that if D, E, F be three points on the sides of a triangle ABC , then the product

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF}$$

is negative or positive according as the lines AD, BE, CF are concurrent or the points D, E, F are collinear, the absolute value of the product in each case remaining the same *viz.*, one. It should be noted that a ratio of the segments of a side is negative or positive according as the point of division lies within or without it. Hence in order that the product be negative, the three ratios must be either all negative, or one negative and two positive so that for *concurrency* the three points D, E, F either all lie within the sides or one within and two without. Similarly, it may be seen that if the three points are collinear, they must either all lie without the sides or one without and two within.

3.8. An alternative proof of Desargues' Theorem on triangles in plane perspective is given below. The proof is based on the two theorems proved above. [For. fig. ref. § 2.11].

Since the points B_1, C_1, P lying on the sides OB, OC and BC of $\triangle BOC$ are collinear, we have

$$\frac{BP}{CP} \cdot \frac{CC_1}{OC_1} \cdot \frac{OB_1}{BB_1} = 1. \quad \dots (1)$$

Similarly, from the triangles OCA, OAB , we have

$$\frac{CQ}{AQ} \cdot \frac{AA_1}{OA_1} \cdot \frac{OC_1}{CC_1} = 1. \quad \dots (2)$$

$$\frac{AR}{BR} \cdot \frac{BB_1}{OB_1} \cdot \frac{OA_1}{AA_1} = 1. \quad \dots \quad \dots (3)$$

From (1), (2), (3), on multiplication, we get

$$\frac{BP}{CP} \cdot \frac{CQ}{AQ} \cdot \frac{AR}{BR} = 1.$$

which shows that the points P, Q, R lying on the sides BC, CA, AB of the $\triangle ABC$ are collinear.

Conversely, let P, Q, R be collinear. Then AA_1, BB_1, CC_1 are concurrent.

Let BB_1, CC_1 meet at O .

We have to show that O, A, A_1 are collinear.

Consider the triangles BB_1R and CC_1Q . The lines BC, B_1C_1, RQ joining the corresponding vertices of these two triangles are concurrent in P . Hence, by the direct theorem, their corresponding sides $BB_1, CC_1; B_1R, C_1Q; RB, QC$ intersect in collinear points. $\therefore O, A_1, A$ are collinear.

Exercises

1. Show that

(i) the medians of a triangle are concurrent.

(ii) the internal bisectors of the angles of a triangle are concurrent.

(iii) the altitudes of a triangle are concurrent.

2. The lines joining the vertices of a triangle to the points of contact with the opposite sides of the ϵ -circles are concurrent.

3. A circle drawn in the plane of the triangle ABC cuts the sides BC, CA, AB respectively in $D, D'; E, E'; F, F'$ and the lines AD, BE, CF are concurrent; prove that AD', BE', CF' are also concurrent.

4. With the help of Menelaus' Theorem, prove that the lines joining the mid-points of two sides of a triangle is parallel to the third.

5. The tangents to the circumcircle at the vertices of a triangle meet the opposite sides in three collinear points.

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6. Three points X, Y, Z are taken on the sides BC, CA, AB respectively of a triangle ABC such that AX, BY, CZ are concurrent; XY, YZ, ZX meet AB, BC, CA in Z', X', Y' respectively; show that

- (i) X', Y', Z' , are collinear,
- (ii) AX, BY', CZ' , are concurrent.

7. The sides BC, CA, AB of a triangle ABC are cut by two lines in the points D, E, F , and D', E', F' ; show that EF', FD', DE' cut BC, CA, AB respectively in collinear points.

8. H, K are the points of trisection nearest B of the sides BC and BA respectively of a triangle ABC ; CK meets AH at P . Prove that $HP/PA = 1/3$.

9. Points Y and Z are taken on the sides CA, AB of a triangle such that $CY = YA$ and $AZ = 2 \cdot ZB$. The lines BY and CZ meet at P . Show that $BP = PY$ and $CP = 3 \cdot PZ$.

10. D and E are the mid-points of the sides BC and CA of a triangle ABC and H, K are the points of trisection nearest B of the sides BC and BA respectively. The line CK intersects AD in P , BP intersects AH in Q and CQ intersects BE in R . Show that R is a point of trisection of BE nearest B .

11. If D, E, F be three points on the sides BC, CA, AB of a triangle, then show that

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = \frac{\sin BAD}{\sin CAD} \cdot \frac{\sin CBE}{\sin ABE} \cdot \frac{\sin ACF}{\sin BCF}$$

and deduce that the necessary and sufficient condition for the lines AD, BE, CF to be concurrent is that

$$\frac{\sin BAD}{\sin CAD} \cdot \frac{\sin CBE}{\sin ABE} \cdot \frac{\sin ACF}{\sin BCF} = -1.$$

$$\left\{ \frac{BD}{CD} = \frac{\Delta BAD}{\Delta CAD} = \frac{\frac{1}{2} \cdot AB \cdot AD \cdot \sin BAD}{\frac{1}{2} \cdot AC \cdot AD \cdot \sin CAD} = \frac{AB \cdot \sin BAD}{AC \cdot \sin CAD} \right\}$$

12. Lines through the vertices of a triangle ABC equally inclined to the bisectors of the angles meet the opposite sides in $D, D', E, E'; F, F'$ respectively. If AD, BE, CF are concurrent, prove that AD', BE', CF' are also concurrent.

[Pair of lines AD, AD' which are equally inclined to the bisector

of the angle between the lines AB, AC are said to be **Isogonal Conjugates** with respect to AB, AC].

13. ABC is a triangle; AD, BE, CF are perpendiculars on the opposite sides. Show that the lines AG, BH, CK drawn perpendicular to EF, FD, DE respectively are concurrent.

✓14. The necessary and sufficient condition that two coplanar triangles ABC and $A'B'C'$ should be in perspective is that

$$BC_1 \cdot BC_2 \cdot AB_1 \cdot AB_2 \cdot CA_1 \cdot CA_2 = BA_1 \cdot BA_2 \cdot CB_1 \cdot CB_2 \cdot AC_1 \cdot AC_2$$

where A_1, A_2 are the points in which BC is cut by the sides $C'A', A'B'$ and so on.

15. On the sides BC, CA, AB of a triangle are taken the points X, Y, Z such that XY is parallel to AB and ZX is parallel to AC ; BY, CZ meet at O , and AO cuts YZ at U . Show that

$$\frac{BX}{XC} = \frac{ZU}{UY}$$

16. $ABCD$ is a quadrangle; the sides AB and CD being parallel. Through the point of intersection E of AC, BD a line EO is drawn of any length parallel to AB . If OC and BD meet in H and OB and CA in K , prove that KD and HA meet on OE .



CHAPTER IV

CROSS RATIOS HOMOGRAPHIC RANGES AND PENCILS.

✓ 4.1. **Cross ratio of a range.** Def. If A, B, C, D , be four collinear points, the expression

$$\frac{AB \cdot CD}{AD \cdot CB}$$



is called a **cross-ratio** of the given range and is denoted by the symbol $(ABCD)$.

The following essentials of the definition should be noted : —

(i) The order of the letters in the symbol $(ABCD)$ is the same as in the numerator.

(ii) Each letter occurs only once in the numerator and the denominator.

(iii) If the letters of the numerator be placed round a circle in the order in which they appear in the numerator in the clockwise direction, then the counterclockwise sense of those letters gives the denominator.

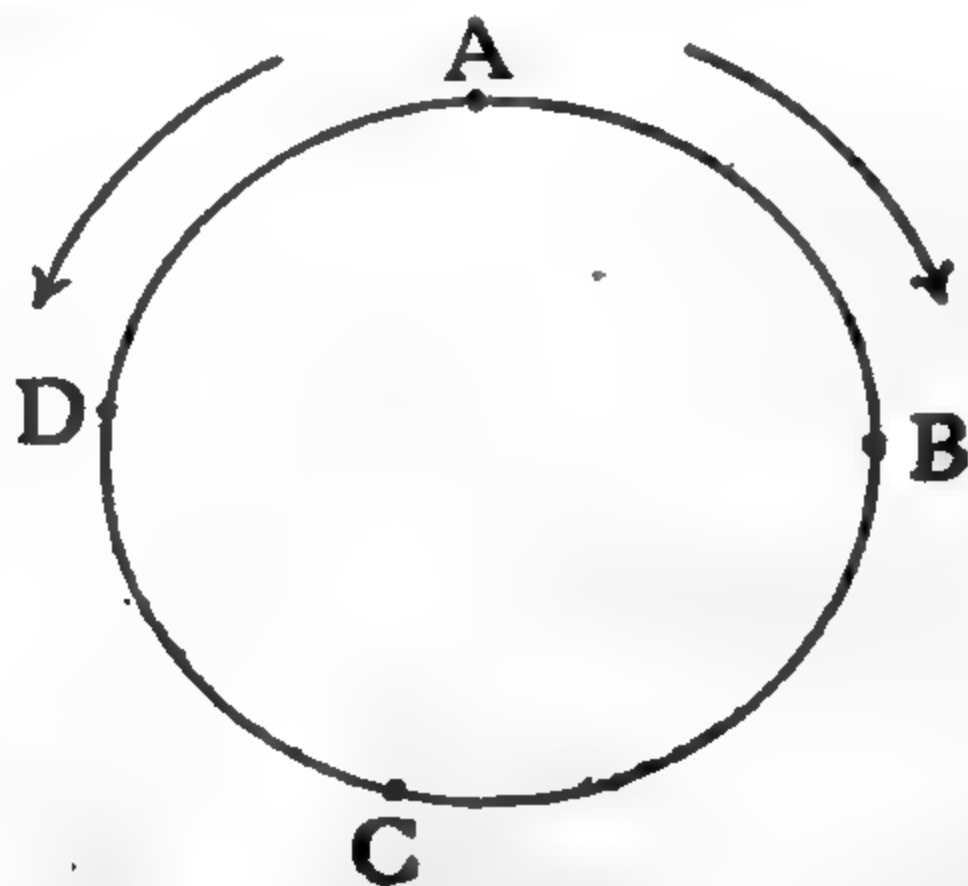


Fig. 36

4.11. Evidently, the cross-ratio of a range of four points depends upon the order in which the points are taken so that, in general, we may obtain as many cross ratios of a range as the number in which the four letters of the range can be arranged among themselves. This number is $4! = 24$. Hence we should expect 24 different cross-ratios in all. It will, however, be shown in the next section that these 24 cross-ratios are divided into six distinct groups of four each and further that these six distinct values are interdependent so that when one of them is known, the rest may be written down at once.

Note 1. The cross-ratio $(ABCD)$ is some times defined as the ratio of the * position ratios of B and D w.r.t. A and C , for

$$(ABCD) = \frac{AB \cdot CD}{AD \cdot CB} = \frac{AB}{CB} \bigg/ \frac{AD}{CD}.$$

On this account, it is also sometimes written as (AC, BD) .

2. A cross-ratio is also sometimes called an *Anharmonic ratio*.

Exercises.

1. Show that $(ABCD) = (BADC) = (CDAB) = (DCBA)$.

2. Show that $(ABCD)(ADCB) = 1$.

3. Prove the following results:—

(i) $(PAQB)(PBQC)(PCQD)(PDQA) = 1$.

(ii) $\frac{(ACPQ)(BDPQ)}{(ADPQ)(BCPQ)} = (ACBD)$.

(iii) $\frac{[(PQRA) - (PQRC)][(PQRB) - (PQRD)]}{[(PQRA) - (PQRD)][(PQRB) - (PQRC)]} = (ABCD)$.

4. If x_1, x_2, x_3, x_4 are the distances of A, B, C, D respectively from a fixed point O lying on the line containing them, then

$$(ABCD) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}.$$

[The expression on the R.H.S. is, for the sake of brevity, denoted as $(x_1 x_2 x_3 x_4)$.]

5. Show that if $x_r = bx_r' + d$, $r = 1, 2, 3, 4$, then

$$(x_1 x_2 x_3 x_4) = (x_1' x_2' x_3' x_4').$$

*The position ratio, of a point P w.r. to A and B is AP/BP .

✓ 6. Show that if $x_r = \frac{cx'_r + d}{ax'_r + b}$, $r=1, 2, 3, 4$, then

$$(x_1 x_2 x_3 x_4) = (x'_1 x'_2 x'_3 x'_4).$$

✓ 7. AD, BE, CF and AD', BE', CF' are two sets of concurrent straight lines drawn through the vertices A, B, C of a triangle to meet the opposite sides in $D, D'; E, E'; F, F'$ respectively; show that

$$(BD CD') \times (CE AE') \times (AF BF') = 1$$

8. Any two transversals cut the sides of a triangle ABC in D, E, F and D', E', F' ; prove that

$$(BD CD') \times (CE AE') \times (AF BF') = 1$$

✓ 42. Theorem. The 24 cross ratios of a range of four points can be divided into six distinct groups of four each, the six values being capable of expression in terms of any one of them.

Firstly, we show that cross-ratio of a range of four points is unaltered if any two points be interchanged, provided the remaining two be also interchanged. i.e., we show that

$$(ABCD) = (BADC) = (CDAB) = (DCBA).$$

$$\text{We have } (BADC) = \frac{BA \cdot DC}{BC \cdot DA} = \frac{AB \cdot CD}{AD \cdot CB} = (ABCD)$$

Similarly, for the remaining two.

Now let $(ABCD) = \lambda$, so that we have

$$(ABCD) = (BADC) = (CDAB) = (DCBA) = \lambda \dots \dots \dots (i)$$

Secondly, we show that a cross ratio is inverted by interchanging the second and fourth letters, or, the first and the third letters. We have

$$(ADC B) = \frac{AD \cdot CB}{AB \cdot CD} = \frac{1}{(ABCD)} = \frac{1}{\lambda}.$$

Hence, by applying the first principle to this cross-ratio,

$$(ADC B) = (DABC) = (BCDA) = (CBAD) = \frac{1}{\lambda} \dots \dots \dots (ii)$$

A similar result will be obtained if we had interchanged the first and the third letters.

Thirdly, we show that an interchange of the second and third letters changes λ into $1 - \lambda$.

By Euler's Theorem, we have

$$AB.CD + AC.DB + AD.BC = 0$$

$$\therefore \frac{AB.CD}{AD.CB} + \frac{AC.DB}{AD.CB} - 1 = 0.$$

$$\text{or} \quad \frac{AC.BD}{AD.BC} = 1 - \frac{AB.CD}{AD.CB}$$

$$\text{or} \quad (\bar{A}CBD) = 1 - \lambda.$$

Then by applying the first principle,

$$(ACBD) = (CADB) = (BDAC) = (DBCA) = 1 - \lambda \dots\dots (iii).$$

From (iii), by interchanging second and fourth letters,

$$(ADBC) = (CBDA) = (BCAD) = (DACB) = \frac{1}{1 - \lambda} \dots\dots\dots (iv)$$

From (iv), by interchanging second and third letters,

$$\begin{aligned} (ABDC) &= (CDBA) = (BACD) = (DCAB) \\ &= 1 - \frac{1}{1 - \lambda} = \frac{\lambda}{\lambda - 1} \dots\dots\dots (v) \end{aligned}$$

From (v), by interchanging the second and fourth letters,

$$(ACDB) = (CABD) = (BDCA) = (DBAC) = \frac{\lambda - 1}{\lambda} \dots\dots\dots (vi)$$

Thus we have classified all the 24 cross ratios in 6 classes and expressed each of them in terms of one of them, λ .

Note 1. If we are given any permutation of the four letters. A, B, C, D , it is always possible to write down another permutation, in A which occupies the first place, the value of the cross-ratio remaining unaffected, for in the given permutation, we have to interchange A with the first element and also interchange

the other two. Thus distinct cross-ratios can be derived from only those permutations, in which A stands first. The number of such permutations being $3! = 6$, we see that we may expect only 6 distinct cross ratios.

This is only another proof of the first part of the theorem.

Note. 2. If $(ABCD) = -1$, the 6 distinct cross-ratios combine in three sets of two each the three values being $-1, \frac{1}{2}, 2$. Conversely, if the value of a cross-ratio is $\frac{1}{2}$ or 2 , an arrangement of the letters certainly exists so that the corresponding cross-ratios is -1 . In this arrangement, the range is said to be *Harmonic*.

Note 3. If $(ABCD) = w$, where w stands for an imaginary cube root of unity, the 6 distinct cross-ratios combine in two sets of three each, the other value being w^2 . The range A, B, C, D , is said to be *bivalent*.

Exercise 4. If two ranges of 4 points are to be equiconjugate their respective cross-ratios are equal.

Exercises.

1. If two points of a range of four points coincide, each of the cross ratios is $0, 1$ or ∞ ; and no cross ratio can equal $0, 1$ or ∞ unless two of the points coincide.

2. Given that

$$(ACDB) = \lambda,$$

find $(BCAD)$ and $(ABCD)$ in terms of λ .

3. If A, B, C, D, E , are five collinear points, and

$$(ACBD) = m, (ACBE) = n;$$

prove that

$$(ADCE) = (n-1)/(m-1).$$

4. If then,

$$(ABCD) = (ABC'D'),$$

$$(ABCC') = (ABDD').$$

5. If

$$(PQRS) = (P'Q'R'S'),$$

then

$$(i) \quad \frac{PQ \cdot RS}{P'Q'} + \frac{PR \cdot SQ}{P'R'} + \frac{PS \cdot QR}{P'S'} = 0$$

$$(ii) \quad \frac{PQ \cdot RS}{P'Q'} \cdot O'Q' + \frac{PR \cdot SQ}{P'R'} \cdot O'R' + \frac{PS \cdot QR}{P'S'} \cdot O'S' = 0;$$

O' being any point on the line $P'Q'$.

have to show that $\frac{PQ \cdot RS}{PS \cdot QR} \cdot \frac{1}{P'Q'} + \frac{PR \cdot SQ}{PS \cdot QR} \cdot \frac{1}{P'R'} + \frac{1}{P'S'} = 0$

$$\begin{aligned} (ABCD) &= 1-m \\ (ABCE) &= 1-n \\ \therefore \frac{AB \cdot CD}{AC \cdot BD} &= \frac{AB \cdot CE}{AC \cdot BE} \\ &= \frac{1-m}{1-n} \\ \therefore (AECB) &= \frac{1-n}{1-m} \\ \text{or } (ADCE) &= \frac{n-1}{m-1} \end{aligned}$$

6. If the circles on AC and BD as diameters intersect at an angle 2θ , prove that

$$(ABCD) = -\tan^2 \theta, \quad (ADCB) = -\cot^2 \theta$$

$$(ACDB) = \operatorname{cosec}^2 \theta, \quad (ABDC) = \sin^2 \theta.$$

$$(ADBC) = \cos^2 \theta, \quad (ACBD) = \sec^2 \theta.$$

4.3. Two important properties of cross-ratios.

4.31. If A, B, C be three different collinear points and D, E two other points collinear with them such that

$$(ABCD) = (ABCE),$$

then D must coincide with E .

Since
$$\frac{AB \cdot CD}{AD \cdot CB} = \frac{AB \cdot CE}{AE \cdot CB}.$$

$$\therefore CD \cdot AE = CE \cdot AD.$$

or $(AD - AC) \cdot AE = (AE - AC) \cdot AD.$

(inserting the origin at A)

or $AE = AD$

so that D and E must coincide, for, A, C being different points, $AC \neq 0$.

Cor. Given a set of three collinear points A, B, C , there is one and only one position of a point D on the line such that the range $(ABCD)$ has a given value.

[This result is of fundamental importance in the study of Analytical Geometry based on a system of homogeneous co-ordinates.]

4.32. If A, B, C be three different collinear points, and X the point at infinity on the line, then

$$(ABCX) = \frac{AB}{CB}.$$

Let P be any point on the line.

$$\therefore (ABCP) = \frac{AB \cdot CP}{AP \cdot CB} = \frac{AB}{CB} \cdot \frac{CP}{AP}$$

$$= \frac{AB}{CB} \cdot \left(\frac{CA + AP}{AP} \right)$$

$$= \frac{AB}{CB} \cdot \left(1 + \frac{CA}{AP} \right) \dots \dots \dots (i)$$

Let $P \rightarrow X$. The point X being at infinity, $AP \rightarrow \infty$ and therefore, $CA/AP \rightarrow 0$. Thus, from (1), we obtain

$$(ABCX) = \frac{AB}{CB}.$$

✓✓ 4.4. Cross ratio of a pencil. Def. If VA, VB, VC, VD form a pencil, the ratio

$$\frac{\sin AVB \cdot \sin CVD}{\sin AVD \cdot \sin CVB}$$

is defined to be a cross ratio of the pencil, (due regard having been paid to the sense of the angles involved), and is denoted by $V(ABCD)$.

If the rays of the pencil be denoted by a, b, c, d , the cross-ratio of the pencil is also denoted as $(abcd)$ or $V(abcd)$.

Ex. If A, B, C, D are any four fixed points on a circle, and P , any variable on the circle, the cross ratio of the pencil $P(ABCD)$ is constant.

✓ 4.1. The following theorem connects the cross-ratio of a pencil with that of a range.

1947 Theorem. The cross-ratio of a pencil is equal to the cross-ratio of the range in which any transversal cuts the pencil.

We are to show that $V(ABCD) = (ABCD)$.

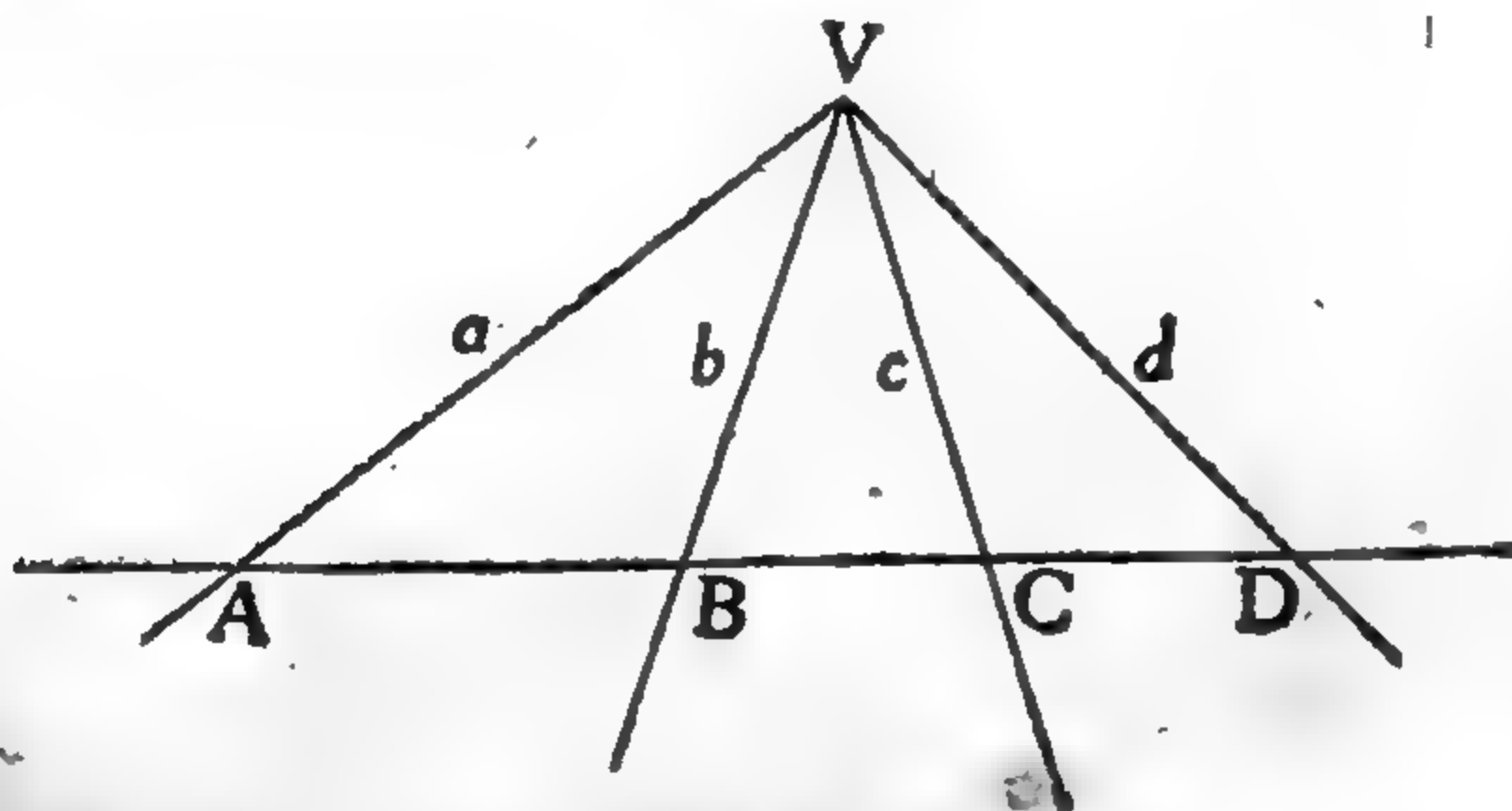


Fig. 40.

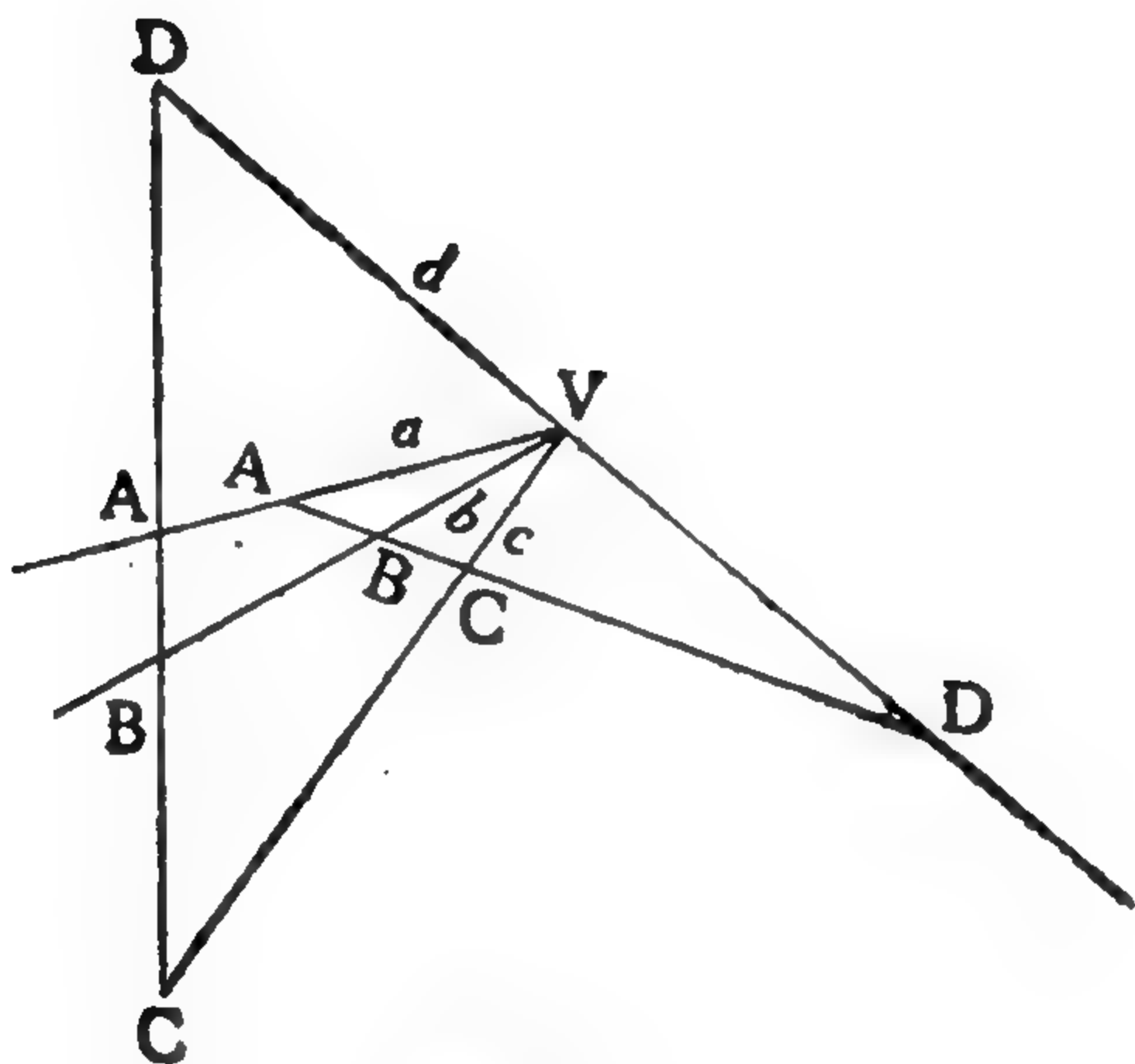


Fig. 41.

We have

$$\begin{aligned}
 (ABCD) &= \frac{AB \cdot CD}{AD \cdot CB} \\
 &= \frac{\Delta AVB \cdot \Delta CVD}{\Delta AVD \cdot \Delta CVB} \\
 &= \frac{\frac{1}{2}VA \cdot VB \cdot \sin AVB \cdot \frac{1}{2}VC \cdot VD \cdot \sin CVD}{\frac{1}{2}VA \cdot VD \cdot \sin AVD \cdot \frac{1}{2}VC \cdot VB \cdot \sin CVB} \\
 &= \frac{\sin AVB \cdot \sin CVD}{\sin AVD \cdot \sin CVB} \\
 &= V(ABCD),
 \end{aligned}$$

paying due regard to the sense of the segments and the angles involved.

The above result can also be stated as follows :—

If two lines cut a pencil in the two ranges A, B, C, D and A', B', C', D' , then $(ABCD) = (A'B'C'D')$.

We have

$$(ABCD) = V(ABCD) = V(A'B'C'D') = (A'B'C'D')$$

4.5. **Equi-cross Ranges and Pencils.** Def. 1. If a cross-ratio of a range of four points equals a cross-ratio of another range of four points, the two ranges are said to be *equi-cross*.

Thus the ranges in which two transversals cut a pencil are equi-cross.

Def. 2. Two *pencils* are said to be equi-cross, if the ranges determined by them on two transversals, one on each, are equicross.

Def. 3. A *pencil* and a *range* are said to be equi-cross when the range is equi-cross with the range determined by the pencil on any transversal.

✓ 4.5. **Theorem.** A range is equi-cross with its projection.

Let the range $(ABCD)$ project into $(A'B'C'D')$ with any point V as the vertex of projection. Then we have to show that

$$(ABCD) = (A'B'C'D').$$

The lines AA' , BB' , CC' , DD' form a pencil with vertex, V , and $ABCD$, $A'B'C'D'$ are two transversals of this pencil. Hence

$$(ABCD) = V(ABCD) = V(A'B'C'D') = (A'B'C'D').$$

Thus the cross-ratio of a range remains unaltered through projection even though it is a metrical concept. A similar result is proved for a pencil in the next section.

4.52 Theorem A pencil is equi-cross with its projection.

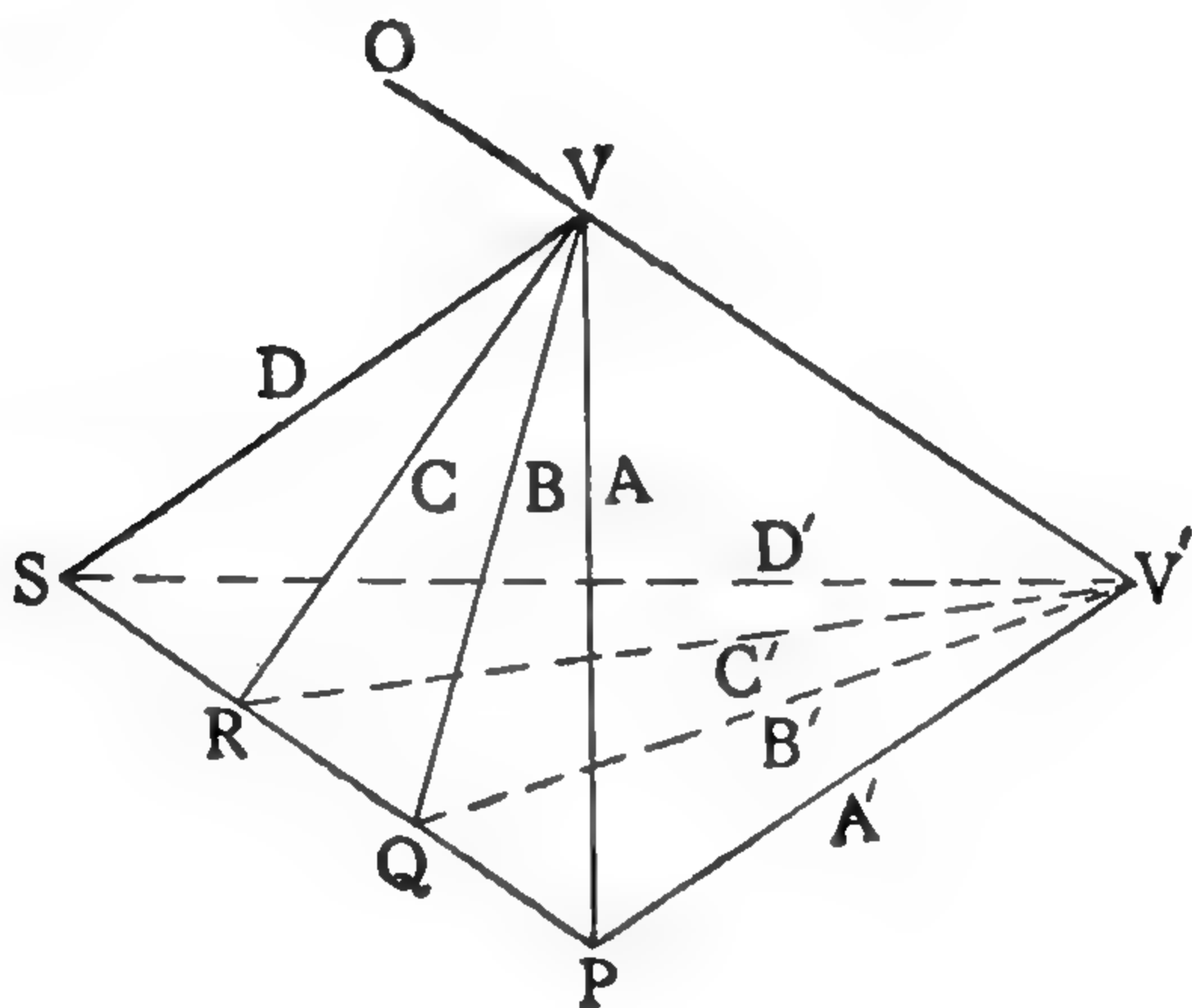


Fig. 42

Let $V(ABCD)$ be a pencil on the α plane and let $V'(A'B'C'D')$ be its projection on the plane of projection π , O , being the vertex of projection.

The corresponding pairs of lines $VA, V'A'$; $VB, V'B'$; $VC, V'C'$; $VD, V'D'$ meet on the axis of projection.

Let their points of intersection be P, Q, R, S , respectively.

Thus P, Q, R, S is a common transversal of the two pencils $V(ABCD)$ and $V'(A'B'C'D')$. Hence

$$V(ABCD) = V(PQRS) = V'(PQRS) = V'(A'B'C'D')$$

4.6. Projective Ranges. In § 2.6 we proved that a range of three points is projective with any other range of three points in space. We also remarked there that a range of four points is not necessarily projective with another range of four points. We shall now lay down the condition when two such ranges are projective.

Theorem. If $[ABCD]$, $[A'B'C'D']$ be two distinct ranges of points such that

$$(ABCD) = (A'B'C'D')$$

then the ranges are projective.

Join AA' and take any point V on it.

The lines VAA' , VB , VC , VD lie in a plane, say α . Through A' draw any line $A'X$ lying in the plane α .

Let the lines VB , VC , VD meet the line $A'X$ in Q , R , S , respectively.

The lines $A'QRS$ and $A'B'C'D'$, being intersecting, are co-planar.

Let the lines QB' and RC' meet in V' . Join $V'D'$ to meet $A'X$ in S' .

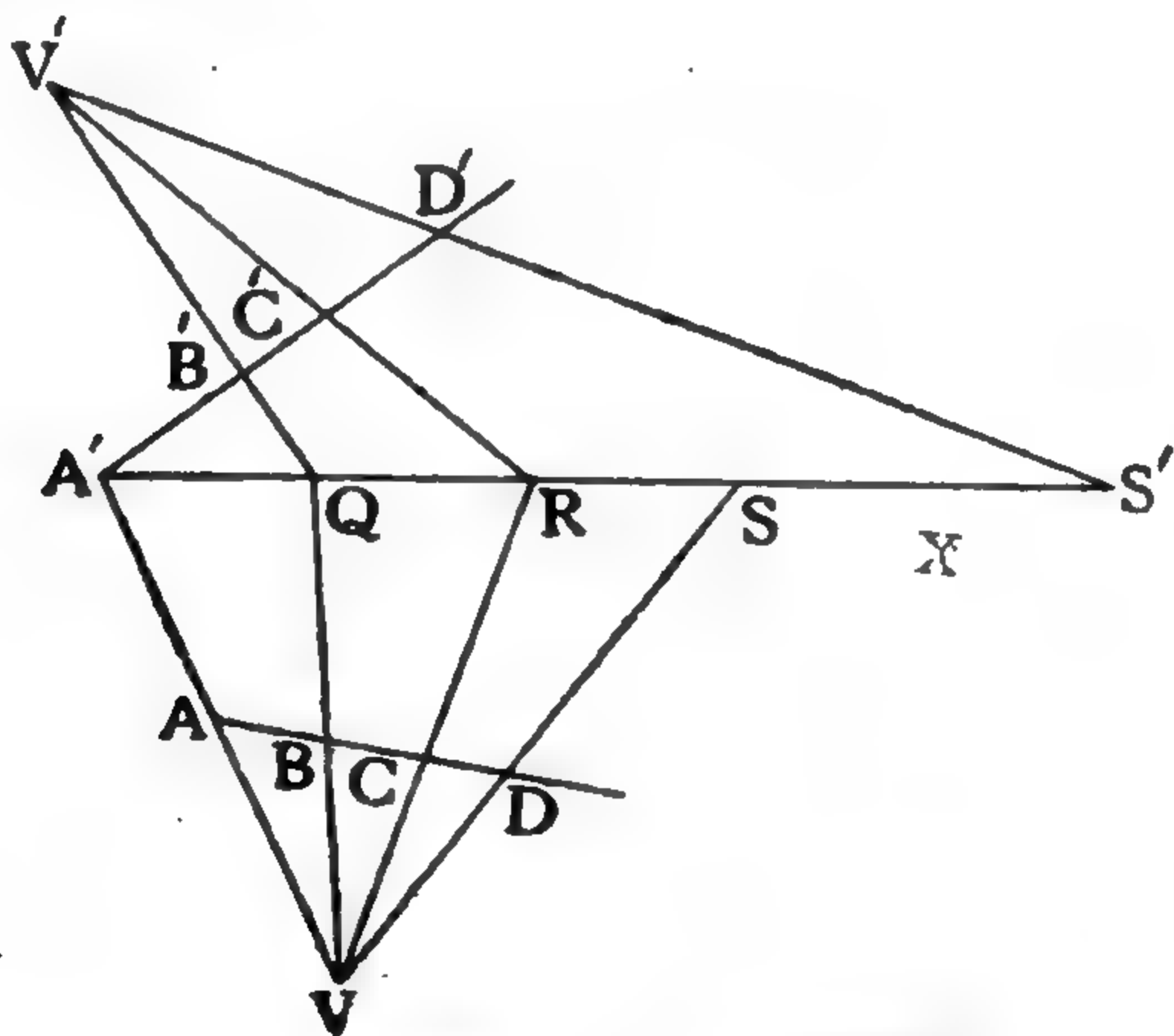


Fig. 43.

Since the range $[A'QRS]$ is the projection of $[ABCD]$ with V as vertex of projection,

$$(ABCD) = (A'QRS) \quad \dots\dots\dots(1)$$

Also, since $[A'QRS']$ is the projection of $[A'B'C'D']$ with V' as the vertex of projection,

$$(A'B'C'D') = (A'QRS') \quad \dots\dots\dots(2).$$

Also, we are given that

$$(ABCD) = (A'B'C'D') \dots\dots\dots(3),$$

$\therefore$ from (1), (2) and (3),

$$(A'QRS) = (A'QRS')$$

Hence S' must coincide with S .

Thus, by means of the vertex V , $[ABCD]$ has been projected into $[A'QRS]$ which, by means of V' , has been projected into $[A'B'C'D']$ so that the two ranges $[ABCD]$ and $[A'B'C'D']$ are projective.

Ex. ABC is any triangle and A_1, B_1, C_1 are any points whatever on the sides BC, CA, AB , show that the expression

$$\frac{BA_1}{CA_1} \times \frac{CB_1}{AB_1} \times \frac{AC_1}{BC_1}$$

is "projective" (i.e., remains unaltered through projection).

Let A_2, B_2, C_2 be the points at infinity on the lines BC, CA, AB . Denoting the projections by corresponding dashed letters, we have

$$\begin{aligned} \frac{BA_1}{CA_1} \times \frac{CB_1}{AB_1} \times \frac{AC_1}{BC_1} &= (BA_1CA_2) \times (CB_1AB_2) \times (AC_1BC_2) \\ &= (B'A_1'C'A_2')(C'B_1'A'B_2')(A'C_1'B'C_2') \\ &= \frac{B'A_1' \cdot C'A_2'}{B'A_2' \cdot C'A_1'} \cdot \frac{C'B_1' \cdot A'B_2'}{C'B_2' \cdot A'B_1'} \cdot \frac{A'C_1' \cdot B'C_2'}{A'C_2' \cdot B'C_1'} \\ &= \left(\frac{B'A_1'}{C'A_1'} \cdot \frac{C'B_1'}{A'B_1'} \cdot \frac{A'C_1'}{B'C_1'} \right) \cdot \left(\frac{C'A_2'}{B'A_2'} \cdot \frac{B'C_2'}{A'C_2'} \cdot \frac{A'B_2'}{B'C_2'} \right) \\ &= \frac{B'A_1'}{C'A_1'} \cdot \frac{C'B_1'}{A'B_1'} \cdot \frac{A'C_1'}{B'C_1'} \end{aligned}$$

for the expression in the second bracket is unity, A_2', B_2', C_2' being three collinear point on the sides of the $\triangle A'B'C'$.

Homographic ranges and Pencils.

4.7. Def. If a rational algebraic (1-1) correspondence exists between the elements of two ranges $[ABCD\dots]$ and $[A'B'C'D'\dots]$, on the same or different bases, the two ranges are said to be homographic. *if every cross*
(The two ranges may lie on the same or different bases.)

every four points (ABCD) of one range is equal

Let O, O' denote the origins for the two ranges. Let P be any element of the first range, and P' the corresponding element of the second. Let

$$OP = x, O'P' = x'.$$

We want, now, to obtain a general analytical relation connecting x and x' so that the two ranges may be homographic.

This will be obtained by considering the following three limitations imposed by the definition of Homography on the correspondence between x and x' .

The correspondence must be

(i) algebraic, (ii) rational, (iii) one-to-one.

Since the correspondence is algebraic, the relation between x and x' must not involve any transcendental functions such as Logarithmic, Trigonometric etc.

As the correspondence is rational, the algebraic relation must not involve any irrationality such as the extraction of roots. It must involve only the four fundamental operations of Addition, Subtraction, Multiplication and Division performed only a finite number of times upon x, x' and constants.

Finally, the correspondence being (1-1), the relation must have both x and x' occurring *linearly* in it, for, the existence of a higher power of any of them, say x , will give more than one value of x corresponding to a given value of x' so that to a point P' in the second range, there will correspond more than one element of the first range which is against the hypothesis.

Thus the most general relationship connecting x and x' must be of the form

$$kxx' + lx + mx' + n = 0, \quad (1)$$

where k, l, m, n are any arbitrary constants and k, l, m are not all zero.

There is no loss of generality if, while studying two co-basal (lying on the same base) homographic ranges, we measure the distances x and x' from the same point i. e., if we choose the same point as origin for the two ranges.

✓ **Note 1.** An important point to note in the case of two homographic ranges on the same base is that the correspondence between the elements is *not* necessarily reversible, i.e., if P , as an element of the first range, corresponds to P' as an element of the second range, then P' as an element of the first range may *not* have P as a corresponding element in the second range. This is so because an interchange of x and x' does not leave the homographic relation necessarily unchanged.

The study of such reversible correspondence will be taken up in Chapter VI.

✓ **Note 2.** The student may note that x and x' connected by a relation such as $x' = \log x$ which is (1-1) but not algebraic, will not generate homographic ranges.

✓ **Note 3.** We re-write the relation (1) in the form

$$x'(kx+m) + (lx+n) = 0 \quad \dots \quad \dots \quad (2)$$

Let the two linear equations

$$kx+m=0, \quad lx+n=0 \quad \dots \quad \dots \quad (3)$$

be consistent i.e., be satisfied by the same value α of x . This is the case if and only if the equations

$$lkx+lm=0, \quad klx+kn=0$$

are consistent which is so if and only if

$$lm=kn. \quad \dots \quad \dots \quad \dots \quad (4)$$

If this condition is satisfied, then for $x=\alpha$, (2) becomes an identity in x' and is, therefore, satisfied by every value of x' . Thus, in this case, there exists a value α of x to which correspond an infinite number of values of x' .

Again, multiplying (2) by l and employing the relation (4), we re-write it in the form

$$kx'(lx+n) + l(lx+n) = 0.$$

If $x \neq \alpha$ i.e., $lx+n \neq 0$, we obtain $kx'+l=0$ so that x' is a constant, say, β , whatever value x may have, excepting α . Thus if $lm=kn$, is

satisfied, we see that there exists also a value β of x' to which correspond an infinite number of x .

Thus, we see that the relation (1) is homographic if and only if the arbitrary constants l, m, n, k satisfy the inequality

$$lm \neq nk.$$

4.71. The homographic relation

$$kxx' + l x + m x' + n = 0$$

contains three effective constants (the ratios of any one of the four letters k, l, m, n to the other three) which can be uniquely determined from three independent equations involving them; it follows that if the relation is true of three pairs of corresponding points, it is true for every pair of corresponding points of the two homographic ranges.

Hence three pairs of corresponding points determine homography.

✓ Exercises.

1. ✓ $A, A'; B, B'; C, C'$ are three pairs of corresponding points on the same base; $OA=1, OB=3, OC=6; OA'=1, OB'=2, OC'=3$.

Prove that, taking O as origin for both, the corresponding homographic relation is $xx' - 7x + 9x' - 3 = 0$.

Find the point of either range corresponding to the point at infinity of the other.

2. ✓ Show that the relation $2xx' - x + 12x' - 6 = 0$ does not determine homography.

3. ✓ Two homographic ranges whose bases are the x -axis and y -axis are defined by $2xy - 3x + 4y - 5 = 0$. Find the points, P', Q corresponding to the origin for the two ranges, also find the points I', J corresponding to the points at infinity of the two ranges.

4. ✓ Through a fixed point O , a variable line is drawn to cut two fixed lines in P and P' , show that P and P' generate homographic ranges.

5. A circle S is drawn so as to touch the two co-ordinate axes.

$$\frac{\{ \text{See } AVB. \text{ See } CVD \}}{A'B'} = 0.$$

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the tangent at any point A of the circle meets the two axes in P and P' , show that P and P' generate homographic ranges.

8. A, B are fixed points, P is a variable point on a fixed line l , AP, BP meet another fixed line m in Q, R . Prove that Q and R describe homographic ranges. Find when Q and R coincide.

47. Theorem. If $[ABCD \dots \dots] [A'B'C'D' \dots \dots]$ be two homographic ranges on the same or different bases, a cross-ratio of any four elements of the one equals the corresponding cross-ratio of the four corresponding elements of the other.

Let the homographic relation be

$$kxx' + lx + mx' + n = 0 \dots \dots \dots (l \ m \neq k \ n) \quad (1.)$$

Let any four elements of the first range be A_1, A_2, A_3, A_4 . Let x_1, x_2, x_3, x_4 be the distances of A_1, A_2, A_3, A_4 respectively from the origin O , and let x'_1, x'_2, x'_3, x'_4 be those of A'_1, A'_2, A'_3, A'_4 respectively from the origin O' .

We have
$$x = -\frac{mx' + n}{kx' + l}$$

$$\text{Now } (A_1 A_2 A_3 A_4) = \frac{A_1 A_2 A_3 A_4}{A_1 A_4 A_3 A_2} = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_1)(x_2 - x_3)}$$

$$= \left[-\frac{(mx'_2 + n)}{(kx'_2 + l)} + \frac{(mx'_1 + n)}{(kx'_1 + l)} \right] \left[-\frac{(mx'_4 + n)}{(kx'_4 + l)} + \frac{(mx'_3 + n)}{(kx'_3 + l)} \right]$$

$$= \left[-\frac{(mx'_4 + n)}{(kx'_2 + l)} + \frac{(mx'_1 + n)}{(kx'_1 + l)} \right] \left[-\frac{(mx'_2 + n)}{(kx'_4 + l)} + \frac{(mx'_3 + n)}{(kx'_3 + l)} \right]$$

$$= \frac{(kn - lm)^2 (x'_2 - x'_1)(x'_4 - x'_3)}{(kn - lm)^2 (x'_4 - x'_1)(x'_2 - x'_3)} = \frac{A'_1 A'_2 A'_3 A'_4}{A'_1 A'_4 A'_3 A'_2}$$

$$= (A'_1 B'_1 C'_1 D'_1)$$

Conversely, let

$(ABC \dots \dots P \dots \dots)$ and $A'B'C' \dots \dots P' \dots \dots$ be two ranges such that the cross ratio of any four elements

of the one is equal to the cross-ratio of the corresponding four elements of the other. and let O, O' be any two fixed points of the ranges. Let the distances of A, B, C , from O be a, b, c , etc., and of A', B', C' , etc., from O' be a', b', c' .

Let $OP = x, O'P' = x'$ where P, P' are two *variable* corresponding elements of the two ranges. We have

$$(ABCP) = (A'B'C'P')$$

$$\therefore \frac{AB \cdot CP}{AP \cdot CB} = \frac{A'B' \cdot C'P'}{A'P' \cdot C'B'}$$

$$\text{i.e.,} \quad \frac{(b-a)(x-c)}{(x-a)(b-c)} = \frac{(b'-a')(x'-c')}{(x'-a')(b'-c')} \quad (1)$$

On simplification, we see that (1) is a relation of the form

$$kxx' + lx + mx' + n = 0,$$

so that P and P' describe homographic ranges.

Note. In virtue of the above property, the two homographic ranges are conveniently denoted as

$$(ABCD \dots \dots \dots) = (A'B'C'D' \dots \dots \dots)$$

Note. The property of homographic ranges proved above is fundamental and is sometimes adopted as a definition of two such ranges.

Ex. $A, A'; B, B'; C, C'$ are three pairs of corresponding points on two bases, $OA = a; OB = b, OC = c; O'A' = a', O'B' = b', O'C' = c'$, and O', O , are the two origins. Prove that the corresponding homographic relation is

$$(b-a)(x-c)/(x-a)(b-c) = (b'-a')(x'-c')/(x'-a')(b'-c')$$

4.14. Theorem. *Two homographic ranges are projective.*

The proof is simple and is left as an exercise for the student who is referred to the § 4. 

Conversely two projective ranges are homographic because through projection the cross-ratio remains unaltered.

The following is an immediate consequence of the projectivity of two homographic ranges.

If two homographic ranges

$$[ABCD...], [A'B'C'D'...],$$

on different bases, be such that the lines joining three pairs of corresponding points A, A' ; B, B' ; C, C' be concurrent in O , then will the join of every pair of corresponding points pass through O .

As a particular case of the above, we have

If two coplanar homographic ranges be such that the point of intersection of their bases corresponds to itself in the two ranges, then the joins of corresponding points of the two ranges are concurrent.

✓ *Homographic Pencils. Def. Two pencils are said to be homographic, when they determine homographic ranges on two transversals, one on each.*

From the definition, it at once follows that if two pencils

$V (a b c d \dots)$ and $V' (a' b' c' d' \dots)$ are homographic, the cross ratio of a pencil of four elements of the one equals the corresponding cross-ratio of the four corresponding elements of the other, i.e.,

$$V (a b c d) = V' (a' b' c' d').$$

In virtue of this, the homographic relationship between two pencils is expressed as

$$V (a b c d \dots) = V' (a' b' c' d' \dots).$$

The following theorems are enunciated for two homographic pencils; their proofs which are simple, being left to the student.

∴ 1. *Two homographic pencils are projective and conversely.*

2. *If in two homographic pencils $V (a b c d \dots)$, $V' (a' b' c' d' \dots)$ with different vertices, the meets of*

three pairs of corresponding rays aa' , bb' , cc' are collinear lying on a line o , then will the meet of every pair of corresponding rays lie on o .

3. If two coplanar homographic pencils be such that the join of their vertices corresponds to itself in the two pencils, then the meets of corresponding rays of the two pencils are collinear.

4.73. Vanishing points of two homographic ranges.

Let the points at infinity on the two ranges be denoted by W , W' . If I in the first range corresponds to W' , and J' in the second range corresponds to W , then I and J' are called the vanishing points of the two ranges.

If the homographic relation be

$$kxx' + lx + mx' + n = 0,$$

we have

$$x = -\frac{mx' + n}{kx' + l} = -\frac{m + \frac{n}{x'}}{k + \frac{l}{x'}} \rightarrow -\frac{m}{k} \text{ when } x' \rightarrow \infty$$

Hence for I , $x = -m/k$.

Similarly, for J' , $x' = -l/k$.

Theorem. If P , P' are two corresponding points in two homographic ranges in which I and I' are the vanishing points, then, $IP \cdot J'P'$ is constant.

Let the homographic relation be

$$kxx' + lx + mx' + n = 0.$$

Let O , O' be the origins in the two ranges.

$$\text{Then } OI = -\frac{m}{k}, O'J' = -\frac{l}{k}.$$

Let P , P' be a pair of corresponding points at distances x and x' from their respective origins.

$$\text{Then } IP \cdot J'P' = (OP - OI)(O'P' - O'J').$$

$$= \left(x + \frac{m}{k}\right)\left(x' + \frac{l}{k}\right).$$

$$= xx' + \frac{l}{k}x + \frac{m}{k}x' + \frac{lm}{k^2}.$$

$$\begin{aligned}
 &= \frac{l}{k} (kxx' + lx + mx' + n) + \frac{lm}{k^2} - \frac{n}{k} \\
 &= \frac{lm - kn}{k^2} \text{ which is constant.}
 \end{aligned}$$

Cor. If the origins be chosen at the vanishing points, then $m=0$, $l=0$, and the homographic relation takes a simple form

$$kxx' + n = 0. \quad \text{or} \quad xx' = -n/k.$$

Conversely, if $IP \cdot J'P'$ be constant where I and J' are fixed point, P and P' describe homographic ranges of which I and J' are the vanishing points.

Choose I and J' as origins. Let $IP = x$, $J'P' = x'$.

Then $xx' = \text{Constant} = -n/k$, say.

$$\text{or} \quad kxx' + n = 0$$

which is a linear relation in x and x' . Hence P, P' generate homographic ranges.

Moreover, $x=0$, when $x' \rightarrow \infty$, i. e., I corresponds to W'
and $x'=0$, when $x \rightarrow \infty$, i. e., J' corresponds to W
so that I and J' are the vanishing points of the two ranges.

4.74. Def. When the elements at infinity on the bases of two homographic ranges are corresponding elements of the ranges, the ranges are said to be **similar**.

In other words I and J' coincide with W and W' for similar ranges.

It is easy to see that the homographic relation in this case takes the form

$$lxx' + mx' + n = 0.$$

4.75. United points of two homographic ranges on the same base.

Def. If in two co-basal homographic ranges, a point corresponds to itself, it is called a **united point** of the two ranges.

A united point is also called a double point or a self-corresponding point.

Let the homographic relation be

$$k x x' + l x + m x' + n = 0$$

where x, x' are the distances of two corresponding points P, P' from the same point—the common origin—which will be called O , or O' , according as it is considered to belong to one or the other range.

For a united point, $x = x'$. Hence the united points of the two ranges are the points corresponding to the roots of the equation

$$k x^2 + (l + m) x + n = 0.$$

This, being a quadratic, we conclude that two co-basal homographic ranges have two united points, real, coincident or imaginary.

4.75. A property of united points. If E, F are the double points of the homographic co-basal ranges $(ABC \dots P \dots)$ and $(A'B' \dots P' \dots)$, then $(EFPP')$ is constant.

From § 4.72 since E and F correspond to themselves, we have

$$(EPFQ) = (EP'FQ')$$

$$\text{i. e.,} \quad \frac{EP \cdot FQ}{EQ \cdot FP} = \frac{EP' \cdot FQ'}{EQ' \cdot FP'}$$

$$\text{i. e.,} \quad \frac{EP \cdot FP'}{EP' \cdot FP} = \frac{EQ \cdot FQ'}{EQ' \cdot FQ}$$

$$\text{i. e.,} \quad (EPFP') = (EQFQ')$$

$$\therefore (EFPP') = (EFQQ').$$

4.752. If two homographic pencils have a common vertex, they will have two and only two united or double rays.

For, they determine on any transversal two homographic cobasal ranges which have two and only two

united points which when joined to the common vertex give the double rays.

A theorem similar to the one of § 4.751 holds also for homographic pencils.

✓ We shall close the chapter with a construction for finding the point D on a given range ABC so that the cross ratio $(ABCD)$ has a given value λ .

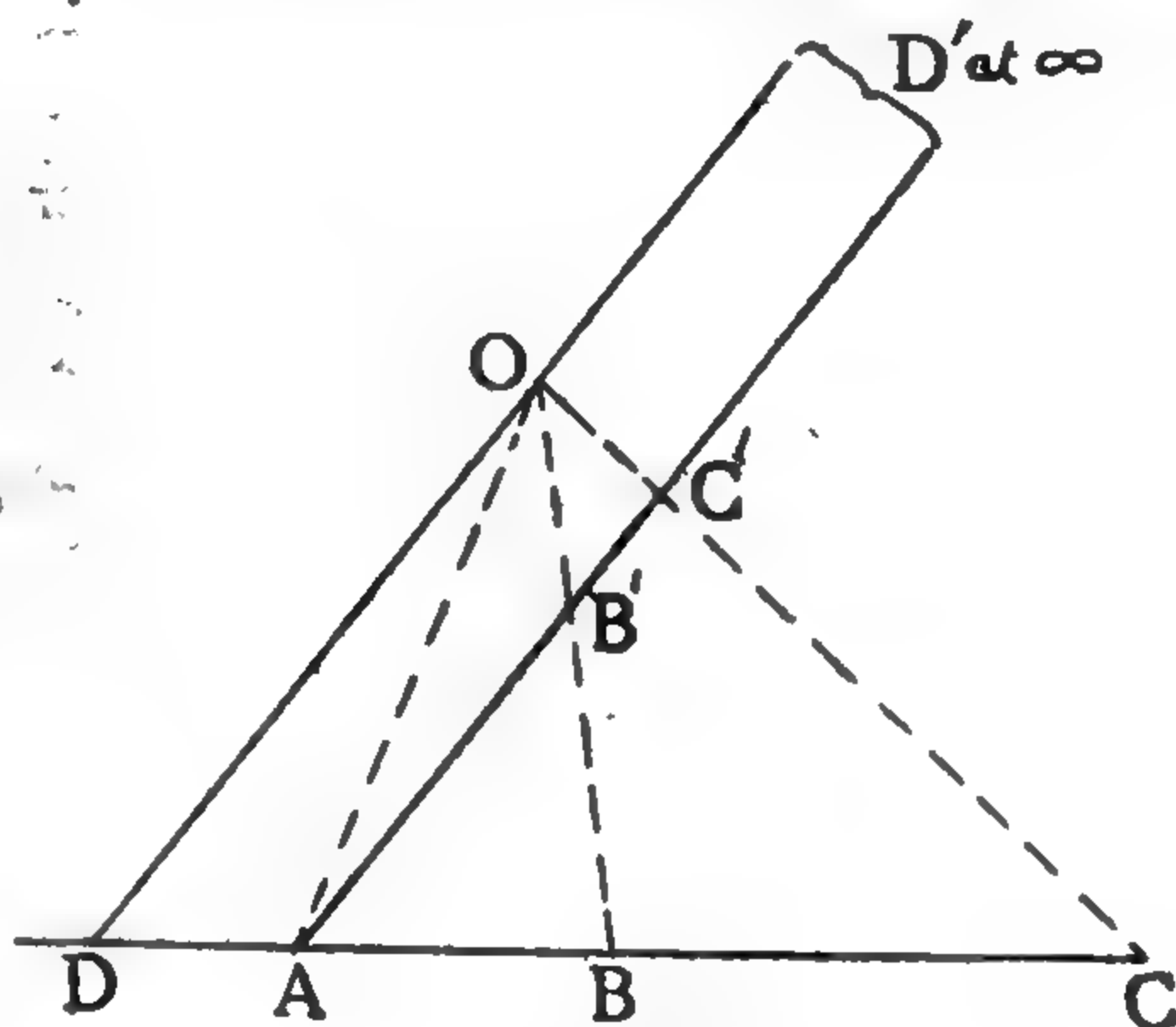


Fig. 44.

Through A draw any line $A B' C'$ and let $AB' = \lambda \cdot C'B'$.

Let BB' , CC' meet in O .

Through O , draw $OD \parallel A B' C'$ meeting ABC in D . Then D is the required point.

For, $(ABCD) = O(ABCD)$.

$= O(AB'C'D')$ where D' is the point at infinity on $AB'C'$.

$$= (AB'C'D') = \frac{AB'}{C'B'} = \lambda.$$

In the diagram attached λ is negative.

Ex. 1. Show that triangles in perspective are co-axal. (Desargue's Theorem.)

In $\Delta s ABC, A'B'C'$, the lines AA', BB', CC' are given to be concurrent in O .

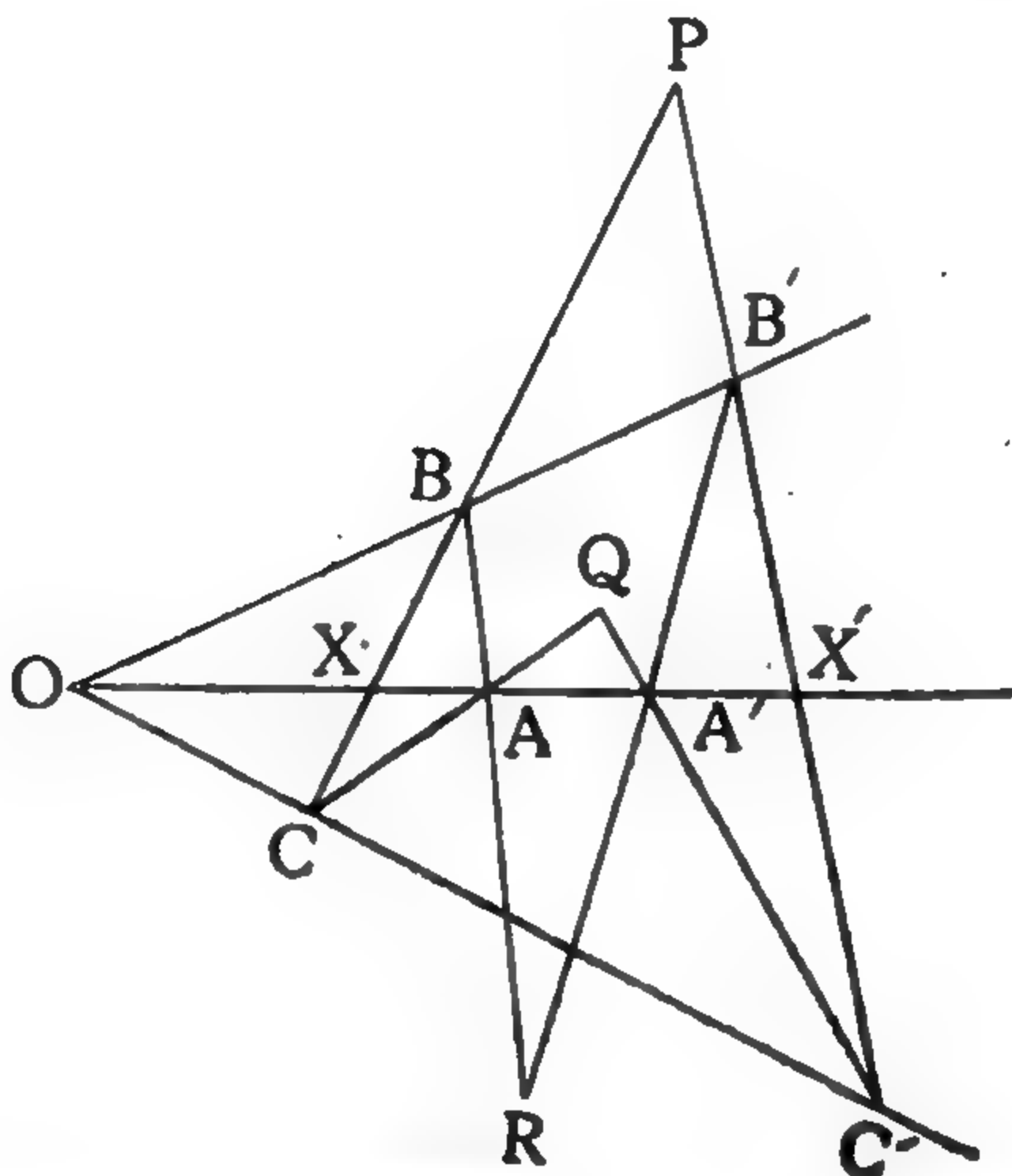


Fig. 45.

Let $BC, B'C'; CA, C'A'; AB, A'B'$ meet in P, Q, R respectively. To prove that P, Q, R are collinear.

Let OAA' meet BC in X , and $B'C'$ in X' .

$$\therefore \begin{aligned} (PBXC) &= O(PBXC) = O(PB'X'C') = (PB'X'C') \\ \therefore A(PBXC) &= A'(PB'X'C') \end{aligned}$$

Therefore the pencils are homographic.

Also they have a common self-corresponding ray $X'AX$.

Hence the meets of corresponding rays BB', CC' lie on the line joining the vertices AA' i.e. P, Q, R are collinear.

Exercises.

1. O is a fixed point and a, b are two fixed straight lines. A point P moves on the line b and Q is taken on the line OP such that $(PORQ)$ is constant, R being the meet of the lines a and OP . Show that the locus of Q is a straight line.

2. For two homographic ranges

$$(ABCD \dots P \dots Q \dots), (A'B'C'D' \dots P' \dots Q' \dots)$$

on different bases, the points of intersection of $PQ', P'Q$ etc., lie on a fixed line.

3. Obtain the Cartesian equation of the

[The feet of the ordinates and abscissae generated by a general conic section are similar to the range of points on the line $ax + my + n = 0$].

CHAPTER V.

HARMONIC FORMS.

5.1. **Harmonic range.** Def. A range of four collinear points A, B, C, D is said to be harmonic if

$$(ABCD) = -1.$$

As pointed out before, the 24 cross ratios of the range divide themselves, in this case, in 3 sets of eight each, the three distinct cross-ratios being -1 , $1/2$ and 2 . We shall, however, reserve the term harmonic for the particular order of the elements corresponding to which the cross-ratio equals -1 .

The set of eight cross-ratios which equal -1 is

$$(ABCD), (BADC), (CDAB), (DCBA), \\ (ADCB), (BCDA), (CBAD), (DABC).$$

5.11. If $(ABCD) = -1$, we have

$$\frac{AB \cdot CD}{AD \cdot CB} = 1.$$



or

$$\frac{AB}{CB} = -\frac{AD}{CD} \quad \dots \dots \dots (1)$$

i.e., B and D divide AC internally and externally in the same ratio.

Similarly, re-writing (1) in the form

$$\frac{BA}{DA} = -\frac{BC}{DC} \quad \dots \dots \dots (2)$$

we observe that A and C divide BD internally and externally in the same ratio.

Conversely, if B and D , (or A and C) divide AC (or BD) internally and externally in the same ratio, then $(ABCD) = -1$.

This follows by working the algebra backward.

Thus, we see that a range of four points is harmonic if and only if the segment determined by two alternate

of members *of* the range is divided by the other two internally and externally in the same ratio.

When $(ABCD) = -1$, we say that the segment AC is *harmonically divided* at B and D and BD is so divided at A and C , or again A and C are said to be *harmonically conjugate* or simply *harmonic*. *w. r. t.* B and D and vice versa, or again we may say that C (or A) is the harmonic of A (or C) with respect to B and D or vice versa. A and C are said to correspond; B and D are said to correspond. The same fact is sometimes symbolically expressed as

$$(AC, BD) = -1.$$

the comma separating pairs of corresponding points $A, C; B, D$.

Note. It will be seen from the set of 8 cross ratios which equal -1 that A and C are alternate members and B and D are alternate.

Properties of a harmonic range.

Prop 5.21. **Theorem.** If $(AC, BD) = -1$, then AC is a harmonic mean between AB and AD , and conversely.

Since $(AC, BD) = -1$ i.e., $(ABCD) = -1$, we have

$$\frac{AB \cdot CD}{AD \cdot CB} = -1.$$

$$\text{or } \frac{AB}{AD} = \frac{BC}{CD} = \frac{AC - AB}{AD - AC}.$$

(Inserting the origin at A)

which shows that AC is a harmonic mean between AB and AD .

On cross multiplying and re-arranging, this gives

$$\frac{2}{AC} = \frac{1}{AB} + \frac{1}{AD}.$$

which gives the same conclusion.

The converse follows by retracing the steps.

Ex. 1. If $(AC, BD) = -1$, show that

$$(i) \quad \frac{2}{BD} = \frac{1}{BA} + \frac{1}{BC} \quad (ii) \quad \frac{2}{DB} = \frac{1}{DC} + \frac{1}{DA}$$

$$(iii) \quad \frac{2}{CA} = \frac{1}{CB} + \frac{1}{CD}$$

Ex. 2. If $(AC, BD) = -1$, and P any other point on AD , show that

$$\frac{2}{AP} = \frac{1}{AB} + \frac{1}{AD}$$

5.22. Theorem. If $(AC, BD) = -1$, and O the mid-point of AC , then $OB \cdot OD = OA^2 = OC^2$ and conversely.

$$\therefore (AC, BD) = -1, \text{ i.e., } (ABCD) = -1,$$

$$\therefore \frac{AB \cdot CD}{AD \cdot CB} = -1.$$

$$\text{or } AB \cdot CD = AD \cdot BC.$$

Inserting the origin at O , this gives

$$(OB - OA)(OD - OC) = (OD - OA)(OC - OB).$$

$$\text{or } (OB - OA)(OD + OA) = (OD - OA)(-OA - OB).$$

$$[\because OA = -OC.]$$

$$\text{or } OB \cdot OD = OA^2.$$

$$\therefore OB \cdot OD = OA^2 = OC^2.$$

Similarly, if O' is the middle point of BD , it can be shown that $O'A \cdot O'C = O'B^2 = O'D^2$.

The converse follows by retracing the steps of the algebra above.

5.3. Theorem. If $(AC, BD) = -1$ and D is the point at infinity on the line then, B is the middle point of AC .

$$\therefore (AC, BD) = -1 \text{ or } (ABCD) = -1, \text{ we have}$$

$$AB \cdot CD = -AD \cdot CB$$



Fig. 46

$$\text{or } -\frac{AB}{CB} = \frac{AD}{CD} = \frac{AC+CD}{CD} = \frac{AC}{DC} + 1$$

Since D is a point at infinity and $AC \neq 0$, therefore, $AC/CD=0$.

Hence $AB = -CB = BC$ i.e. B is the middle point of AC .

The theorem is sometimes enunciated thus :

The fourth harmonic of the middle point of a segment is the point at infinity on the line.

The result could easily be deduced from the 2nd theorem of § 5.2 also. For,

$$OB \cdot OD = OA^2$$

gives $OD = \frac{OA^2}{OB}$

and since O and B coincide i.e. $OB=0$, $\therefore OD \rightarrow \infty$

Cor. The point at infinity on any line bisects every segment of the line externally.

5.4. If $(AC, BD) = -1$, and O be any other point on the line, then

$$2(ac + bd) = (a + c)(b + d)$$

where $OA = a$, $OB = b$, etc.

$$\text{We have } \frac{AB \cdot CD}{AD \cdot CB} = -1.$$

Inserting the origin at O , we have

$$\frac{(OB - OA)(OD - OC)}{(OD - OA)(OB - OC)} = -1.$$

$$\text{or } \frac{(b - a)(d - c)}{(d - a)(b - c)} = -1.$$

$$\text{or } bd - bc - ad + ac = -bd + cd + ab - ac.$$

HARMONIC FORMS

$$\text{or } 2(ac + bd) = ab + bc + ad + cd \\ = (a + c)(b + d) \dots \dots \dots (i).$$

Some deductions.

The theorems of the previous three sections easily come out of this result. *e.g.* let O be at A , so that $a = 0$. The result reduces to

$$2bd = c(b + d).$$

$$\text{or } \frac{2}{c} = \frac{1}{b} + \frac{1}{d} \text{ i.e. } \frac{2}{AC} = \frac{1}{AB} + \frac{1}{AD}.$$

Again, if O be taken as the middle point of AC , $a = -c$.

$$\therefore (-a^2 + bd) = 0.$$

$$\text{or } bd = a^2 = c^2 \text{ i.e. } OB \cdot OD = OA^2 = OC^2,$$

Another important result can be deduced from (i).

Let $a = b$ which means that A and B coincide.

Then (i) gives.

$$2(ac + ad) = (a + c)(a + d),$$

$$\text{or } 2ac + 2ad = a^2 + ac + ad + cd.$$

$$\text{or } a^2 - ac - ad + cd = 0 \text{ i.e. } (a - c)(a - d) = 0,$$

$$\therefore c = a \text{ or } d = a,$$

i.e. either C coincides with A or D coincides with A .

Similar results will follow if we make any other pair of points coincide. Hence the

5.41. Theorem. *If two points of a harmonic range coincide, a third point also coincides with them and the fourth may be anywhere in the line.*

Exercises.

1. $ABCD$ is a rectangle and (AC, PQ) and (BD, XY) are harmonic ranges, show that the four points P, Q, X, Y are concyclic.

2. A straight line meets the sides BC, CA, AB of a triangle ABC at D, E, F ; D', E', F' are the harmonic conjugates of D, E, F with respect to BC, CA, AB respectively; prove that AD', BE', CF' are concurrent.

3. AD, BE, CF are concurrent lines through the vertices of a triangle meeting the opposite sides in D, E, F ; EF, FD, DE meet BC, CA, AB in D', E', F' respectively, prove that

$$(BC, DD') = -1, (CA, EE') = -1, (AB, FF') = -1.$$

PURE GEOMETRY

4. Through a given point O in the plane of a given triangle, draw a line cutting its sides BC, CA, AB in A', B', C' such that $(OA', B'C')$ is harmonic.

✓ 5. P is any point in the plane of a triangle ABC and PD, PE, PF bisect the internal angles formed by the lines joining P to the angular points of the triangle, D, E, F lie on BC, CA, AB respectively, D', E', F' are the harmonic conjugates of D, E, F with respect to B and C, C and A, A and B ; prove that D', E', F' are collinear.

6. ABC is a triangle; A, B, C is a transversal cutting BC, CA, AB at A_1, B_1, C_1 ; A_2, B_2 are two points such that $(BC, A_1A_2) = -1$; $(CA, B_1B_2) = -1$. If AA_2, BB_2 meet at O and CO meets AB at C_2 , prove that C_2 is the harmonic conjugate of C_1 w. r. t. A and B .

7. If (AB, CD) and $(AB', C'D')$ are harmonic ranges on different lines, show that the lines BB', CD', DC' are concurrent.

8. A, B, C are three collinear points in any order whatsoever; show how to project so that C' is the mid point of $A'B'$ where A', B', C' are the projections of A, B, C respectively.

9. Show that it is possible to project so that any two points in two coplanar segments project into the mid-points of the projections of the segments.

10. Prove that a triangle can be so projected that three given concurrent lines through its vertices project into the medians of its projection.

11. Show how to project four non-collinear points in a plane into the vertices and the orthocentre of an equilateral triangle.

✓ 12. Show that the condition for the two pairs of points given by $ax^2 + 2bx + c = 0, a'x^2 + 2b'x + c' = 0$ to be harmonic is $ac' + a'c = 2bb'$.

✓ 13. If A, B are harmonically conjugate w. r. t. C, D and O is the mid-point of AB , prove that

$$(i) \quad \frac{OD}{OB} = \frac{BD}{CB} = \frac{AD}{AC} = \frac{OB}{OC}$$

$$(ii) \quad AC \cdot BD = CD \cdot OB.$$

55. Harmonic Pencil. Def. Four concurrent lines VA, VB, VC, VD are said to form a harmonic pencil if

$$V(ABCD) = -1.$$

Since the cross ratio of a range in which a transversal cuts a pencil is the same as the cross ratio of the pencil, *every transversal cuts a harmonic pencil in a harmonic range.* Conversely, if one transversal l cuts a pencil in a harmonic range, every other transversal will cut it in a harmonic range and the pencil will be harmonic.

Thus a pencil obtained by joining an external point to the points of a harmonic range is harmonic.

Rays of a harmonic pencil which cut a transversal in a pair of conjugate points are said to be *conjugate rays*. The pairs of conjugate rays are said to correspond to each other.

If $V(ABCD) = -1$, then $VA, VC; VB, VD$ are two pairs of conjugate rays. As in the case of a range, this fact is symbolically expressed as

~~$V(ABCD) = -1$~~ , $V(AC, BD) = -1.$

Theorem. The arms of an angle are harmonically conjugate with respect to its internal and external bisectors.

Let VB, VD be the internal and external bisectors of the angle AVC . We are to show that

$$V(AC, BD) = -1.$$

$\therefore VB$ bisects AVC internally,

$$\therefore \frac{AB}{CB} = -\frac{VA}{VC} \quad \dots(1)$$

$\therefore VD$ bisects AVC externally,

$$\therefore \frac{AD}{CD} = \frac{VA}{VC} \quad \dots(2)$$

From (1) and (2), we obtain

$$\frac{AB \cdot CD}{AD \cdot CB} = -1.$$

$$(i.e. AC, BD) = -1.$$

$$\text{Hence } V(AC, BD) = (AC, BD) = -1.$$

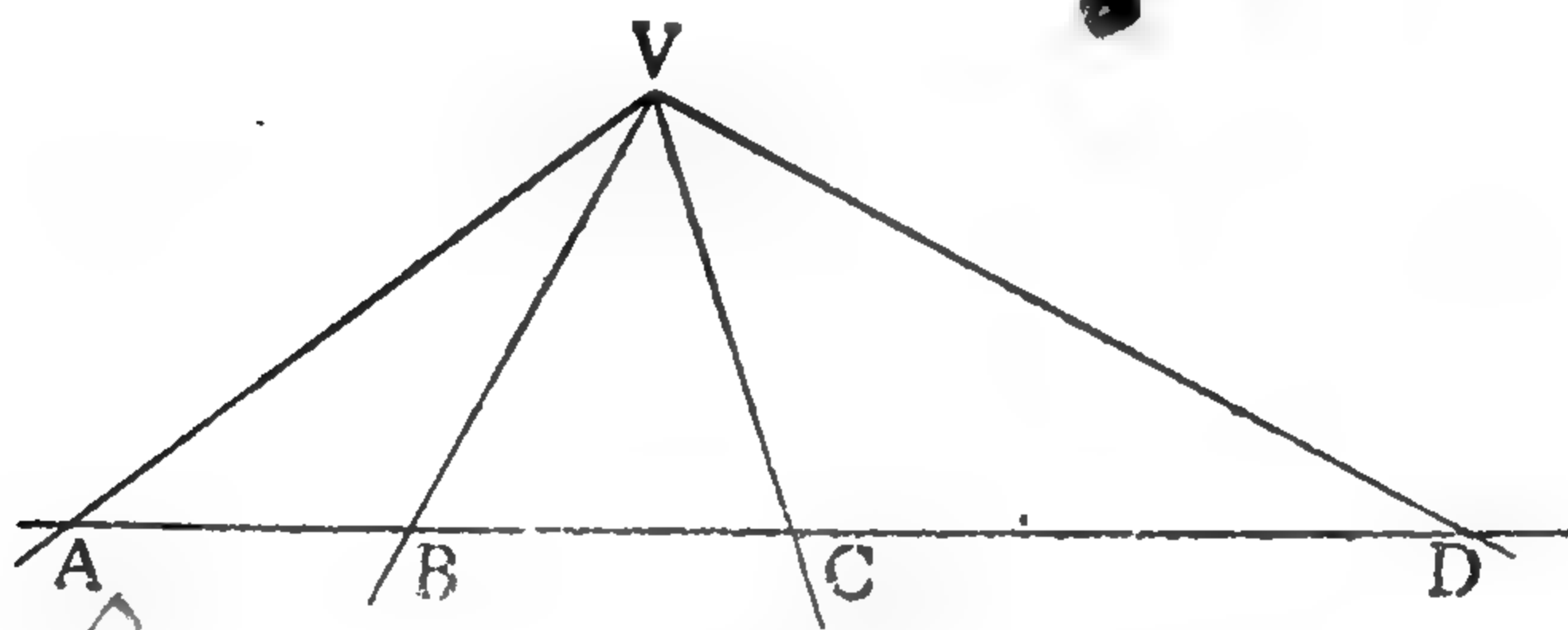


Fig. 47

The converse of the above is stated and proved as follows :—

Theorem. *If a pair of conjugate rays of a harmonic pencil are perpendicular, they are the internal and external bisectors of the angle between the other pair.*

$$\text{Let } V(AC, BD) = -1,$$

and let VB, VD be, the perpendicular conjugate rays of the pencil. We are to show that VB, VD are the internal and external bisectors of the $\angle AVC$.

$$\text{Now } V(ABCD) = \frac{\sin \angle AVB \cdot \sin \angle CVD}{\sin \angle AVD \cdot \sin \angle CVB}$$

Since $\angle BVD = \text{one right angle}$, therefore

$$\sin \angle CVD = \sin (90^\circ - \angle BVC) = \cos \angle BVC$$

$$\text{and } \sin \angle AVD = \sin (90^\circ + \angle AVB) = \cos \angle AVB$$

$$\begin{aligned} \therefore V(ABCD) &= \frac{\sin \angle AVB \cdot \cos \angle BVC}{\cos \angle AVB \cdot (-1) \sin \angle BVC} \\ &= - \frac{\tan \angle AVB}{\tan \angle BVC} \end{aligned}$$

$$\text{But } V(ABCD) = -1,$$

$$\therefore \frac{\tan \angle AVB}{\tan \angle BVC} = 1, \text{ or } \angle AVB = \angle BVC.$$

i. e. VB bisects $\angle AVC$ and VD , being perpendicular to VB , is the other bisector of $\angle AVC$.

5.6. Since the cross ratio of a range or a pencil is unaltered by projection, we observe that

a harmonic range projects into a harmonic range,

a harmonic pencil projects into a harmonic pencil.

5.7. Harmonic properties of the complete quadrangle and the complete quadrilateral.

The dual character of the two figures was specially emphasised while explaining their nature. We give below two theorems of fundamental importance relating to the figures. The dual nature of the two theorems should be carefully noted. The theorem concerning the complete quadrangle is taken first.

Imp 5.7. Theorem. A pair of sides of the triangle formed by the diagonal points of a complete quadrangle is harmonically conjugate with respect to the pair of sides of the quadrangle meeting at the same diagonal point.

Let A, B, C, D be four points forming the quadrangle.

Let E, F, G , be the diagonal points. We are required to show that each of the pencils

$E(AD, FG), F(BA, GE), G(CD, EF)$

is harmonic.

$E(AD, FG), F(CD, GE), G(BA, EF)$

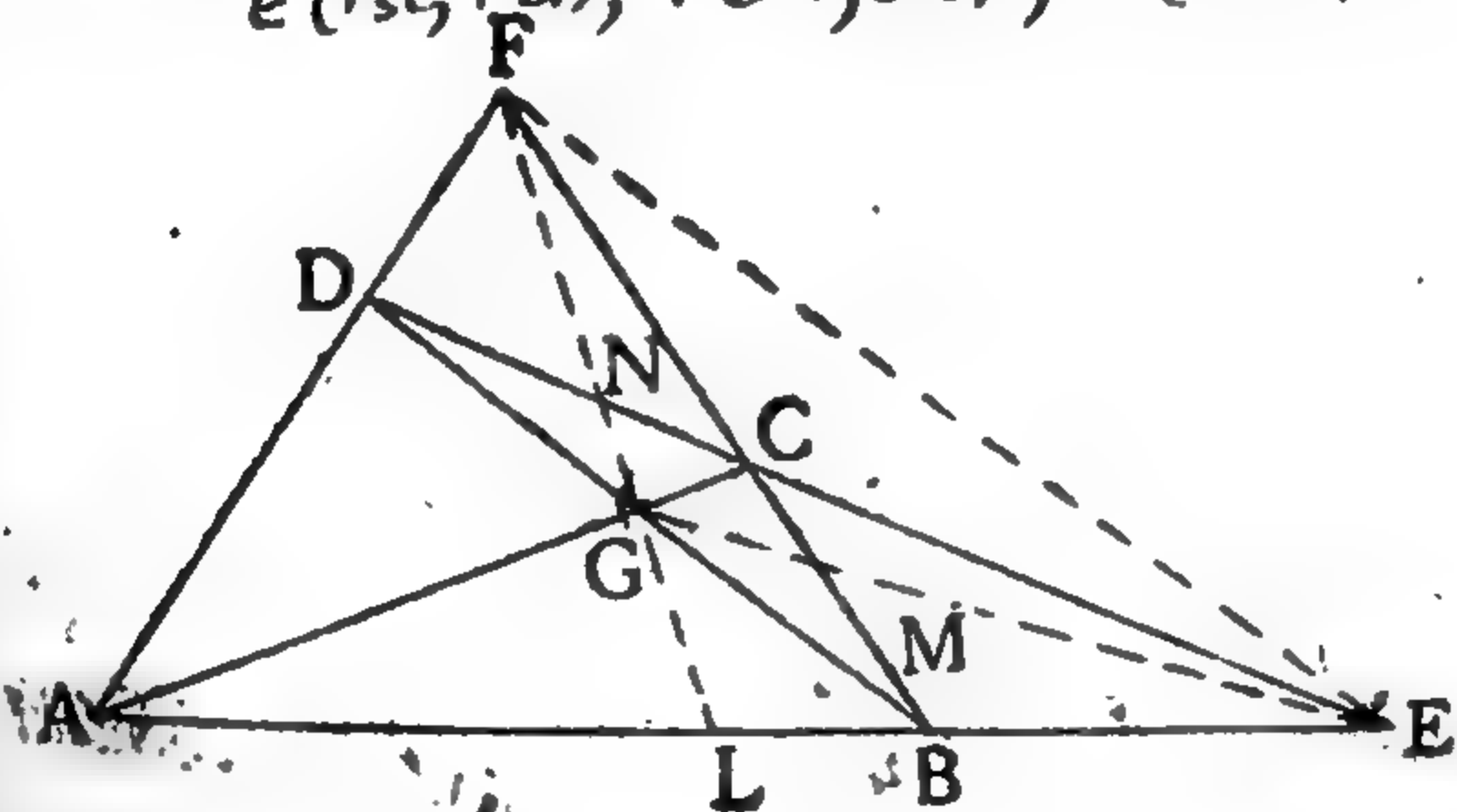


Fig. 48

Consider any one of the pencils, say, $G(CD, EF)$.

To show that $G(CD, EF) = -1$, it is sufficient to show that the range determined by it on the transversal AB , i.e., $(AB, EL) = -1$.

In $\triangle AFB$, the points D, C, E on the sides AF, FB, BA respectively are collinear. Hence applying Menelau's Theorem,

$$\frac{AD}{FD} \cdot \frac{FC}{BC} \cdot \frac{BE}{AE} = 1. \quad \dots (1)$$

Again, in the same triangle, the lines AC, FL, BD drawn through the vertices A, F, B respectively are concurrent (in G),

Hence applying Ceva's Theorem,

$$\frac{AD}{FD} \cdot \frac{FC}{BC} \cdot \frac{BL}{AL} = -1 \quad \dots (2)$$

From (1) and (2) we get

$$\frac{BE}{AE} = -\frac{BL}{AL}$$

i.e., L and E divide BA internally and externally in the same ratio. Thus

$$(AB, EL) = -1. \checkmark$$

Hence $G(CD, EF) = -1$.

Again, $F(BA, GE) = (BA, LE) = (AB, EL) = -1$

and $E(AD, FG) = (BC, FM) = G(BC, FM) = (BA, LE) = (AB, EL) = -1$.

Thus the theorem is proved.

Note. On account of this harmonic property, the diagonal points are called the *harmonic points* of the quadrangle and diagonal triangle is called its *harmonic triangle*.

The following theorem gives the harmonic property of the complete quadrilateral.

5.12 Theorem. A pair of vertices of the triangle formed by the diagonal lines of a complete quadrilateral is

harmonically conjugate with respect to the pair of vertices of the quadrilateral lying on the same diagonal line.

Let $ABCD$ be a quadrilateral formed by the lines AB, BC, CD, DA . Let PQR be the triangle formed by the diagonal lines AC, BD, EF . We are to show that each of the three ranges

$$(AC, QR), (BD, RP), (EF, PQ)$$

is harmonic.

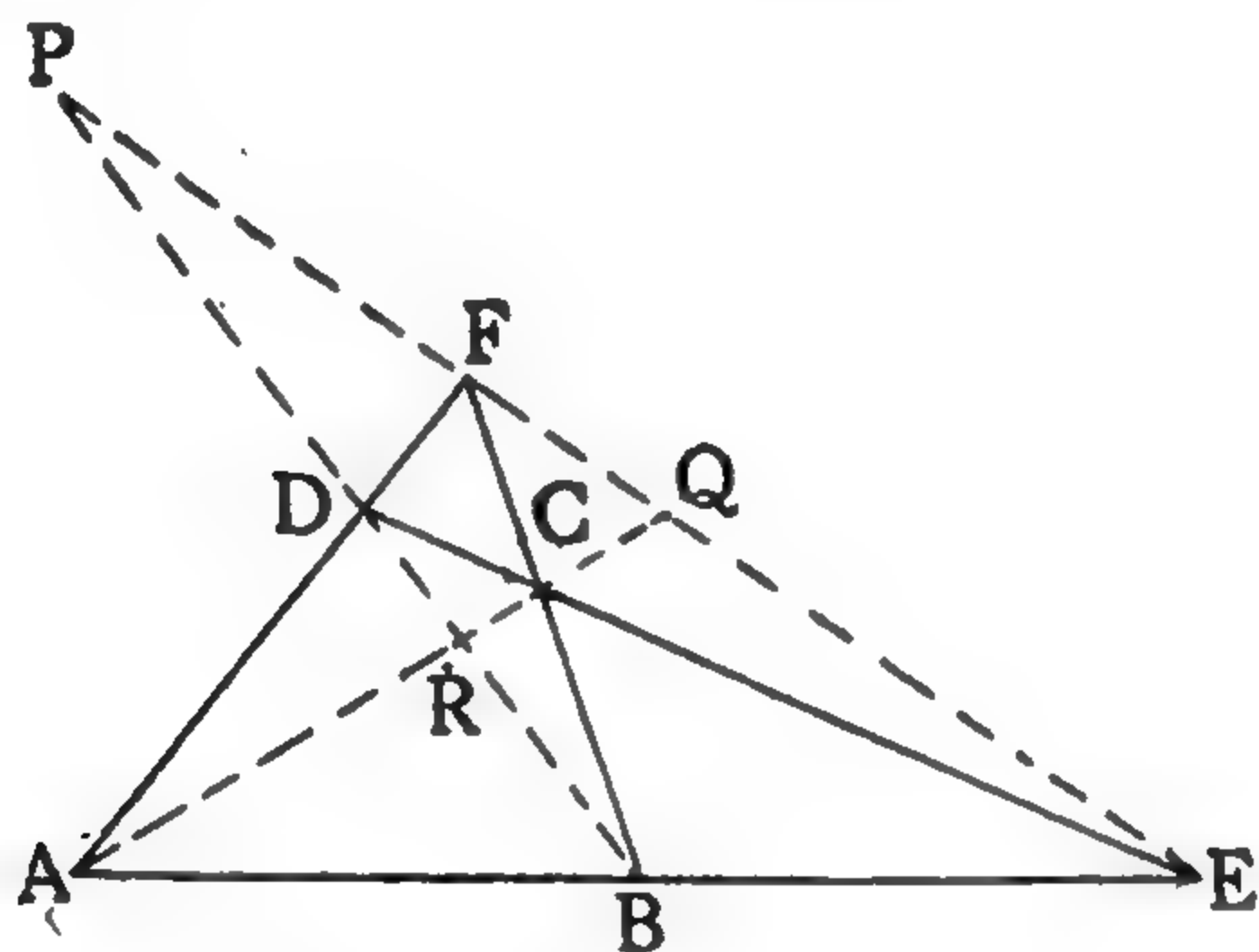


Fig. 49.

To prove that (AC, QR) is harmonic, consider the triangle ACF .

The points B, R, D , on the sides FC, CA, AF of the triangle are collinear. Hence applying Menelaus' Theorem, we have

$$\frac{AR}{CR} \cdot \frac{CB}{FB} \cdot \frac{FD}{AD} = 1 \quad \dots (1)$$

Again, in the same triangle the lines AB, FQ, CD through the vertices A, F, C are concurrent (in E).

Hence applying Ceva's Theorem,

$$\frac{AQ}{CQ} \cdot \frac{CB}{FB} \cdot \frac{FD}{AD} = 1 \quad \dots (2)$$

From (1) and (2), we obtain

$$\frac{AR}{CR} = -\frac{AQ}{CQ}$$

so that R and Q divide AC internally and externally in the same ratio. Thus

$$(AC, QR) = -1.$$

$$\begin{aligned} \text{Again } (BD, RP) &= F(BD, RP) = (CA, RQ) \\ &= (AC, QR) = -1. \end{aligned}$$

$$\text{And, } (EF, PQ) = A(EF, PQ) = (BD, RP) = -1.$$

Hence the theorem is proved.

Note. The triangle PQR i.e. the triangle formed by the diagonal lines of a quadrilateral, is called the **harmonic triangle** of the quadrilateral.

Ex. 1. Establish the theorems of § 5.7 by projection EE to infinity.

Ex. 2. Deduce the harmonic property of the complete quadrilateral from that of the complete quadrangle and *vice versa*.

58. Given a ruler only, to find

(i) the fourth harmonic of given line w. r. t. two given lines,

(ii) the fourth harmonic of given point w. r. t. two given points. (P. 11-46)

(i) Suppose FA, FG, FB are the given rays and we are required to construct the fourth harmonic of FG , w. r. t. FA and FB . (See Fig. 48. P. 75.)

Through G , draw any two lines AGC, DGB , meeting FA in A and D and FB in B and C . Let AB and DC meet in E . Then FE is the required ray; for considering the quadrangle $ABCD$, we at once observe that FG, FE being two sides of the harmonic triangle are conjugate w. r. t. FA, FB , the two sides of the quadrangle through F .

(ii) Suppose A, R, C are the given points and we are required to construct the fourth harmonic of R w. r. t. A and C . (See Fig. 49, P. 77.)

Through R draw any ray, and on it take two points B and D . Let AB, CD meet in E and AD, BC in F . Let AC meet EF in Q . Then Q is the required point. For, considering the quadrilateral $ABCD$, we observe that Q, R the vertices of the harmonic triangle of the quadrilateral, are conjugate w. r. t. A, C , the vertices of the quadrilateral collinear with the former pair.

59. The methods of the previous section for the construction of the fourth harmonic of three rays or three points are based on the harmonic properties of the complete quadrangle and the complete quadrilateral. We suggest alternative methods below.

1. Given three collinear points ABC , to find the point D so that $(AC, BD) = -1$.

Through A draw any line $AB'C'$ and make $AB' = B'C'$. Let BB', CC' meet in V . Through V draw VD parallel to $AB'C'$ meeting ABC in D . Then D is the required point. For, if D' is the point at infinity on $AB'C'$, we have

$$(AC', B'D') = -1,$$

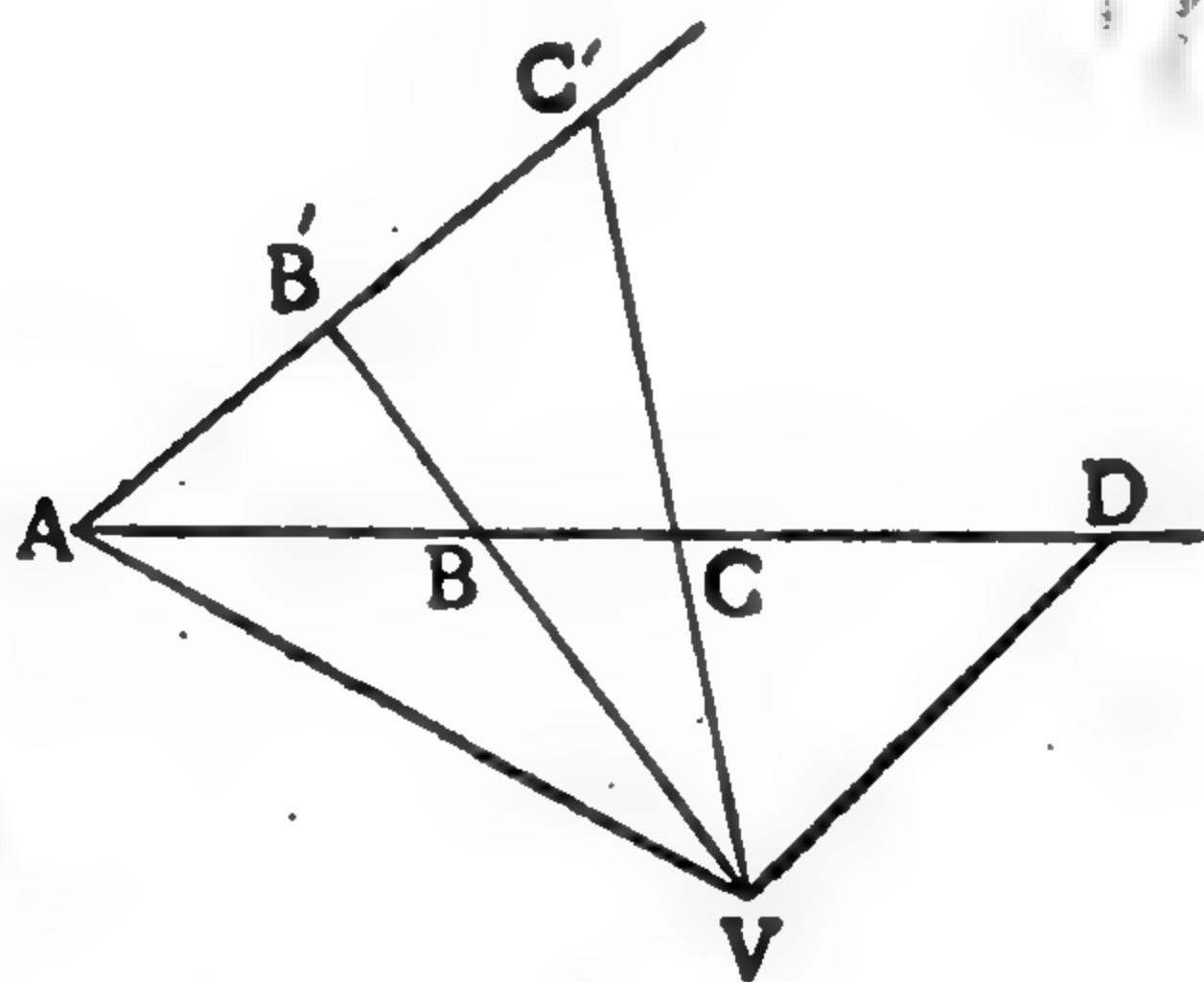


Fig. 50.

B' being the mid-point of AC' .

$$\begin{aligned} \text{Now } (AC, BD) &= V(AC, BD) \\ &= V(AC, BD) \\ &= (AC', B'D') \\ &= -1. \end{aligned}$$

2. Given three concurrent lines VA, VB, VC , to find the ray VD so that $V(AC, BD) = -1$.

The ray VD is to correspond to VB . Draw any line parallel to VB meeting VA, VC in A' and C'

respectively. Let D be the middle point of $A'C'$. Then VD is the required ray.

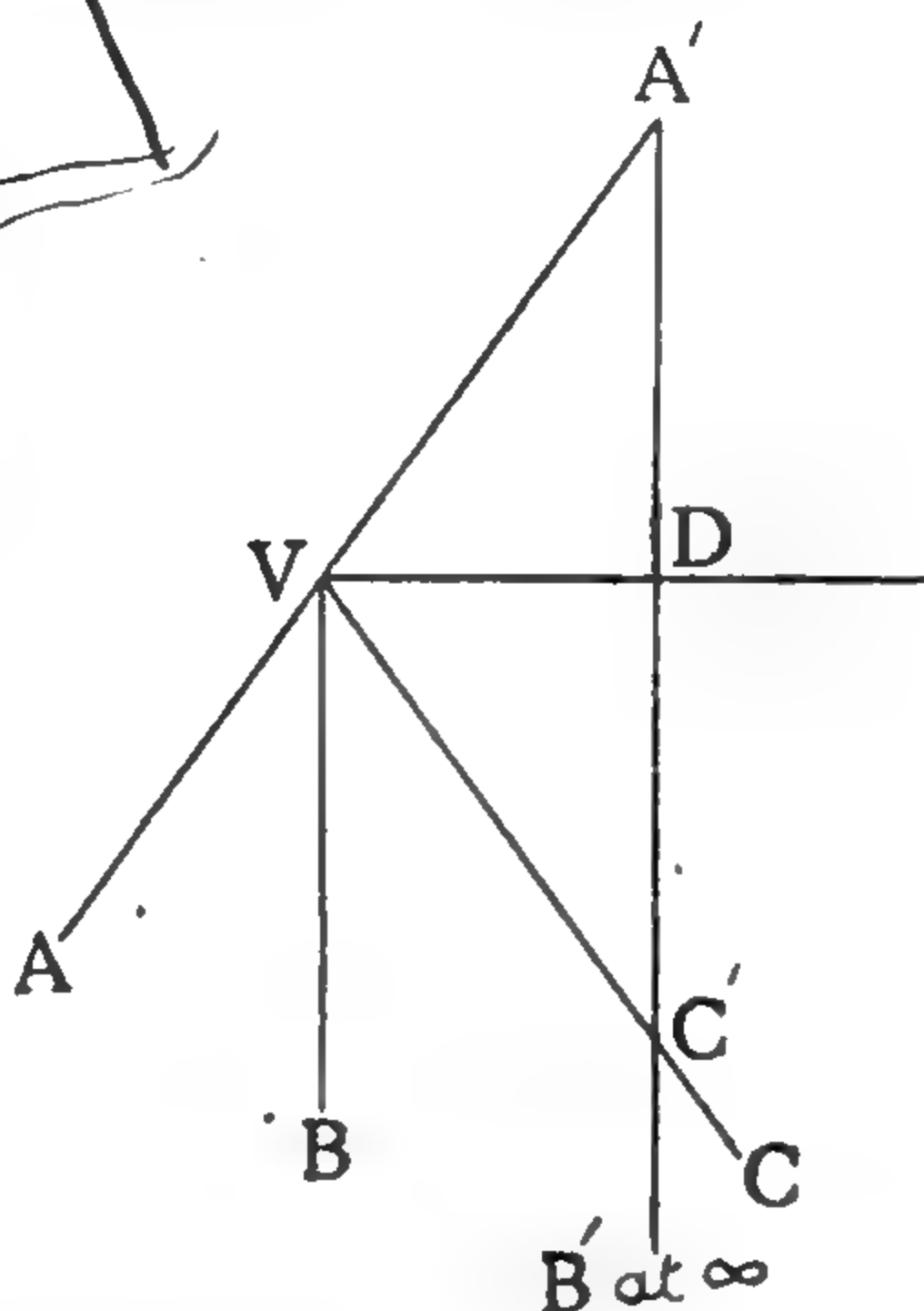


Fig. 51.

Since B' is the point at infinity on $A'C'$, and D is the middle point of $A'C'$, we have

$$V(AC, BD) = (A'C', B'D) = -1.$$

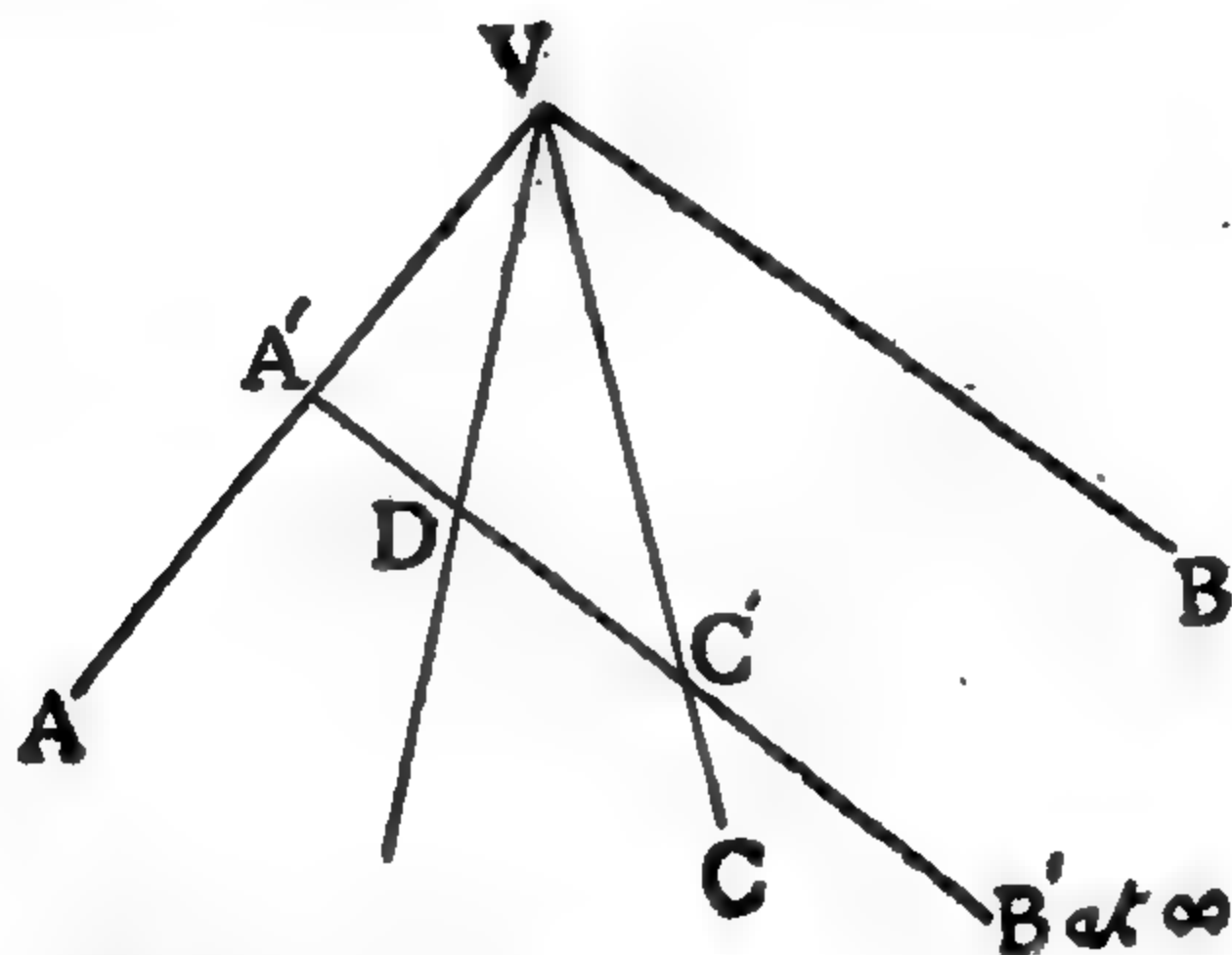


Fig. 52.

Exercises.

1. A, B are two fixed points and l a fixed line. If P be any point on l and p be harmonically conjugate to l with respect to PA, PB , show that p passes through a fixed point.

2. A, B, C are three collinear points; find the locus of a point at which the segments AB and BC subtend equal angles.

3. ABC is a triangle: $D, D'; E, E'; F, F'$ are harmonic conjugates with respect to $B, C; C, A; A, B$ respectively. Prove that the corresponding sides of the triangles DEF and $D'E'F'$ intersect on the sides of the triangle ABC .

4. A and B are two fixed points and l is a fixed straight line; P_1 and P_2 are any two variable points on the line l ; L is the intersection AP_2, BP_1 and M is the intersection of AP_1, BP_2 ; show that for different positions of the points P_1 and P_2 on the line l , the point of intersection of the lines AB and LM remains fixed. use MS
use B

5. A, C, D are three fixed collinear points and DP is a fixed line through D ; G is any variable point on DP . The lines AP, CG meet in Q ; the lines CP, DQ meet in H , and the lines CD, GH meet in R ; prove that R remains fixed while the point G moves along the line DP .

6. ABC is a triangle. On BC two points U, V are taken harmonically conjugate with respect to B, C . If P is any point of AB and VP, PU meet AC at S and R respectively, prove that US and VR meet on AB at a point Q harmonically conjugate of P with respect to AB .

7. If through O , the intersection of the diagonals of a quadrilateral $ABCD$, a line OH be drawn parallel to the side AB meeting CD in G and the third diagonal in H , prove that OH is bisected at G .

8. AB and CD intersect in U ; AC, BD in V ; UV intersects AD and BC in F and G ; BF intersects AC in L ; prove that the range $(ALVC)$ is harmonic and that LG, CF, AU are concurrent.

9. Four circles pass through the same two points, A, B . If any transversal through A cuts the circles in a harmonic range, show that the tangents at B form a harmonic pencil.

10. $PQRS$ is a complete quadrangle having A, B, C as the vertices of its diagonal point triangle, lying on PQ, PR, PS respectively. AB cuts QR in V and PS in U ; show that PV and UR meet in AC .

CHAPTER VI

RANGES AND PENCILS IN INVOLUTION.

✓ 6.1. **Def. Ranges in involution.** *If a (1, 1) rational, algebraic, reversible correspondence exists between the elements of two copasal ranges the two ranges, are said to be in Involution.*

Since a (1, 1) rational algebraic correspondence exists between the elements, the two ranges in involution are also homographic. We see, therefore, that an involution is the result of a *particular* type of homography in which the correspondence is *reversible* i.e., in which to a point P , corresponds the same point, P' , to whichever range P is supposed to belong to.

Two ranges in involution must necessarily be on the same base. Corresponding elements P, P' are said to be *point-pairs in the involution*. They are also said to be *conjugates* and each member of a pair of conjugates is said to be a *mate* of the other.

✓ 6.11. **General Analytical relation which determines ranges in involution.** We take any point O , lying on the common base, as origin for both the ranges. Let P, P' be corresponding elements of the two ranges and let

$$OP = x, OP' = x'.$$

In the first instance, the relation between x and x' which is (1 – 1), rational and algebraic, is of the form

$$kxx' + lx + mx' + n = 0; \quad lm \neq kn.$$

We will now determine the condition so that the relation between x and x' is *reversible* i. e., to any point a on the base there corresponds the same point, say b , whichever range a may belong to. Thus

when $x=a$, $x'=b$ and when $x'=a$, $x=b$ for all values of a .

$$\therefore kab + la + mb + n = 0, kab + lb + ma + n = 0$$

$$\therefore (l-m)(a-b) = 0$$

But $a \neq b$, for in this case a and b are the roots of the quadratic $kx^2 + (l+m)x + n = 0$ which cannot be true for every value a of x .

$$\therefore l = m$$

Again if $l=m$, the relation becomes

$$kxx' + l(x+x') + n = 0$$

$$\text{so that for } x=a, x' = -(al+n)/(ka+l)$$

$$\text{and for } x'=a, x = -(al+n)/(ka+l)$$

so that the relation is reversible.

Thus, we see that two cobasal ranges, referred to the same origin, are in involution if and only if they are determined by the relation

$$kxx' + l(x+x') + n = 0, \quad l^2 \neq kn.$$

Note 1. The fact that $(A, A'); (B, B'); (C, C')$ etc., are point-pairs of an involution, is sometimes expressed by writing that $(A, A'; B, B'; C, C' \dots)$ is an involution range.

Note 2. The student should clearly understand the essential distinction between two ranges in involution and between two cobasal homographic ranges.

Each point of the base can be thought of as belonging to either range for homography as well as for involution. For two homographic ranges, a point of the base gives rise in general to two different points according as it is regarded as belonging to one or the other range but for ranges in involution these two points *always* coincide.

6'12. Since two effective constants appear in the most general involutory relationship, it follows that an involution can be set up by two pairs of corresponding points.

6.13. **United Points.** If we put $x=x'$ in the involutory relation, we get the quadratic $kx^2 + 2lx + n = 0$. Corresponding to the two roots x_1, x_2 of this equation, we have two points U_1, U_2 , each corresponding to itself. These points are called the *united points* (also called *double points* or *foci*) of the involution.

An involution is said to be *hyperbolic*, *parabolic* or *elliptic* according as the united points are real and distinct, coincident or imaginary.

Note 1. The student should clearly understand that U_1 is not a mate of U_2 or vice versa. Each corresponds to itself. That is why they have been denoted by U_1 and U_2 and not by U and U' .

2. Since an involution can be set by two pairs of corresponding points and a united point is equivalent to a pair of coincident corresponding points, an involution can ultimately be set up if we are given (i) a united point and a pair of corresponding points or (ii) both the united points.

6.14. **Def.** The mate of the point at infinity of a range in involution is called the *centre* of the involution. (The reason of this nomenclature will appear in § 6.22).

Exercises.

✓ For the involution relation, $xx' - (x+x') - 3 = 0$, find the points corresponding to $x = -2, -1, 1, 0, 2, 3, \infty$.

Point out the peculiar nature of the points $x = -1$ and $x = 3$. Is there any other such point?

✓ Determine the involution relation determined by the point pairs $(1, \frac{5}{2}); (5, \frac{7}{2})$.

Find out the double points.

✓ Show that the involution determined by the point pairs $(a, b); (c, d)$ is

$$\begin{vmatrix} xx' & x+x' & 1 \\ ab & a+b & 1 \\ cd & c+d & 1 \end{vmatrix} = 0.$$

points are given by $x^2 - 4x - 12 = 0$.

$$(1) \quad xx' - 2(x+x') - 12 = 0$$

INVOLUTION

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Q. $x=a$ is a double point and (c, d) is point pair of an involution, show that the involution is given

$$\begin{vmatrix} xx' & x+x' & 1 \\ a^2 & 2a & 1 \\ cd & c+d & 1 \end{vmatrix} = 0.$$

Q. Deduce the involution relation when the double points are given by $x=a, x=c$.

Q. Determine the involution relation if the double points are given by $ax^2 + 2bx + c = 0$.

6.21. Theorem. Every pair of corresponding elements of an involution is harmonically conjugate w. r. t. its united points.

Let the involutory relation be

$$kxx' + l(x+x') + n = 0 \quad \dots \quad (1)$$

x, x' being the distances of a corresponding point-pair P, P' from a fixed origin.

The united points U_1, U_2 are given by

$$kx^2 + 2lx + n = 0 \quad \dots \quad (2)$$

If x_1, x_2 be the roots of (2), we have

$$x_1 + x_2 = -\frac{2l}{k}, \quad x_1 x_2 = \frac{n}{k}, \quad \dots \quad (3)$$

Re-writing the involutory relation (1) as

$$xx' + \frac{l}{k}(x+x') + \frac{n}{k} = 0$$

and substituting for l/k and n/k in terms of x_1 and x_2 , from (3), we obtain

$$xx' - \frac{x_1 + x_2}{2} (x + x') + x_1 x_2 = 0.$$

or

$$2(x_1 x_2 + xx') = (x_1 + x_2)(x + x')$$

which shows P, P' are harmonically conjugate w. r. t. U_1, U_2 . But P, P' is any pair of corresponding points belonging to the involution. Therefore every pair of corresponding points of an involution is harmonically conjugate w. r. t. its united points.

To prove the converse.

Let any point O on the line be taken as origin. Let

$$OU_1 = a, OU_2 = b, OP = x, OP' = x'.$$

Since

$$(U_1U_2, PP') = -1,$$

we have

$$2(ab + xx') = (a + b)(x + x') \quad (\text{Ref. § 5.4, P. 70})$$

or $2xx' - (a + b)(x + x') + 2ab = 0$,
which shows that P, P' determine an involution.

6.22. Theorem. *The centre of an involution bisects the segment joining the united points.*

Let O be the centre and U_1, U_2 the united points of the involution and let X be the point at infinity on the line. Then, by the previous theorem,

$$(U_1U_2, OX) = -1,$$

and since X is the point at infinity, O must be the middle point of U_1U_2 . *M. 5.3.*

6.23. *If $AA', BB', CC', \dots$ be pairs of points in an involution of which O is the centre, then*

$$OA.OA' = OB.OB' = OC.OC' = \dots$$

and conversely.

Let U_1, U_2 be the united points of the involution. Then since $(U_1U_2, AA') = -1$, and O is the middle point of U_1U_2 , therefore

$$OA.OA' = OU_1^2 = OU_2^2.$$

Similarly, $OB.OB' = OU_1^2 = OU_2^2$; etc.

Hence $OA.OA' = OB.OB' = \dots = OU_1^2 = OU_2^2$.

Conversely, if $OP.OP' = c$, a constant, P, P' generate a range in involution.

Inserting the origin at V , we have

$$(VP - VO)(VP' - VO) = c$$

$$\text{or} \quad (x-l)(x'-l)=c$$

$$\text{or} \quad xx'-l(x+x')+(l^2-c)=0$$

which is an involutory relation.

This property is also sometimes taken as a definition of pairs of points in involution.

Note. The united points will be seen to be real or imaginary according as pairs of corresponding points lie on the same side of the centre or one on either side of it, and conversely.

6'3. The following fundamental property of a range in involution is a direct consequence of the definition.

Let $[A, A' : B, B' : C, C' : \dots\dots\dots]$ be an involution so that $A, A' : B, B' : C, C' : \text{etc.}$, are the point pairs.

Then a range consisting of any number of elements is homographic with the corresponding range determined by their mates.

$$\text{i. e., } (AA'BB'CC' \dots U_1U_2) = (A'AB'BC'C \dots U_1U_2).$$

This follows from the fact that the element of the second range corresponding to the element A of the first is A' , and if A be regarded as belonging to the second, then the point corresponding to it in the first range is A' and so on for $B; B'$ etc.

gmp 6'31. The necessary and sufficient condition that three pairs of points $A, A' : B, B' : C, C'$ may belong to an involution is

$$(AA'BC) = (A'AB'C').$$

The condition is necessary. Let C, C' belong to the involution determined by $A, A' : B, B'$. Then

$$(AA'BB'CC') = (A'AB'BC'C).$$

Hence, in particular

$$(AA'BC) = (A'AB'C').$$

The condition is sufficient. Given that

$$(AA'BC) = (A'AB'C'), \dots \dots \dots (1)$$

we have to show that $A, A' ; B, B' ; C, C' ;$ are in involution.

We know that an involution is determined by two pairs of points. Hence if C' is not the mate of C in the involution determined by $A, A' ; B, B'$, let C'' be its mate.

Then, since $A, A' ; B, B' ; C, C''$ are in involution.

$$\therefore (AA'BC) = (A'AB'C'') \dots \dots (2)$$

$\therefore$ from (1) and (2),

$$(A'AB'C') = (A'AB'C'').$$

$\therefore C''$ coincides with C' .

Note. It should be observed that of the four elements of the cross-ratio on one side of the condition obtained above for three point-pairs to be in involution three have been selected by taking one from each pair, and the fourth is the mate of *any* one of these three, and the cross-ratio on the other side is then formed by the mates of the four elements already selected.

The order of the elements of the cross-ratio on one side, is immaterial but *once these elements are selected and arranged*, the corresponding elements as well as their arrangement in the cross ratio on the other side is then definitely fixed. Thus we could also had

$$(BAB'C) = (B'A'BC')$$

or

$$(B'C'AB) = (BCA'B') \text{ and so on.}$$

2. The above theorem can be generalised into a general theorem which may be enunciated thus : *If in two coplanar homographic ranges, one pair of corresponding elements is in (1, 1) reversible correspondence, every such pair is so and the two ranges determine an involution.*

$$\text{i. e., if } (AA'BCD\dots\dots) = (A'AB'C'D'\dots\dots),$$

$$\text{then } (AA'BB'CC'\dots) = (A'AB'BC'C'\dots\dots).$$

6.32. Theorem. *If $A, A' ; B, B' ; C, C'$ be three pairs of points in involution, then*

$$AB'.BC'.CA' = -A'B.B'C.C'A$$

and conversely.

We have $(AA'BB'CC') = (A'AB'B\acute{C}C')$.

$\therefore (AB'BC') = (A'BB'C).$

or $\frac{AB'.BC'}{AC'.BB'} = \frac{A'B.B'C}{A'C.B'B}$

or $AB'.BC'.CA' = -A'B.B'C.C'A.$

The converse follows by working the algebra backward.

Ex. With the same premises, establish the following relations :

(i) $AB'.BC.C'A' = -A'B.B'C'.CA.$

(ii) $AB.B'C'.CA' = -A'B'.BC.C'A.$

(iii) $AB.B'C.C'A' = -A'B'.BC'.CA.$

into **6.4. Theorem.** A range in involution projects into a range in involution.

Let $A, A_1; B, B_1; C, C_1 \dots$ be pairs of points in involution so that

$$(AA_1BB_1CC_1 \dots) = (A_1AB_1BC_1C \dots)$$

Denoting the projections of the elements by corresponding dashed symbols and observing that homographic ranges project into homographic ranges, we have

$$(A'A_1 B'B_1 C'C_1 \dots) = (A'_1 A' B'_1 B' C'_1 C' \dots)$$

Thus $A', A'_1; B', B'_1; C', C'_1 \dots$ are in involution. This proves the theorem.

6.51. Involution Pencils. If $\vee A, \vee A'; \vee B, \vee B'; \vee C, \vee C';$ etc., are a system of line-pairs drawn through a point $\vee$ such that one (and therefore every) transversal is cut by them in an involution range, then the line-pairs are said to form an involution pencil, and we write that

$$\vee[A, A'; B, B'; C, C'; \text{etc.},]$$

is an involution pencil.

Clearly, two members of any line-pair of an involu-

tion pencil cut a transversal in two points which constitute a point-pair of the involution range on the transversal.

The *united rays* (also called *double rays*) of an involution pencil are the lines which cut a transversal (and therefore every transversal) in a pair of points which are the united points of the involution range determined on the transversal.

6.52. Theorem. *The united rays of an involution pencil are harmonically conjugate w. r. t. every line-pair of the pencil.*

This follows from the definition of united rays.

If $V[A, A' ; B, B' ; C, C', \dots]$ is an involution pencil, then

$$V(AA'BB'CC' \dots) = V(A'AB'BC'C \dots).$$

Let any transversal cut the line-pairs in the point-pairs $A, A' ; B, B' ; C, C'$ etc., so that $(A, A' ; B, B' ; C, C' ; \dots)$ is an involution range, and therefore

$$(AA'BB'CC' \dots) = (A'AB'BC'C \dots)$$

Hence the theorem.

6.53. *An involution pencil is uniquely determined by two line-pairs drawn through the same point.*

This easily follows from the corresponding result on involution range.

6.54. *The necessary and sufficient condition that three line-pairs, $VA, VA' ; VB, VB' ; VC, VC'$ may belong to an involution, is that*

$$V(AA'BC) = V(A'A'BC').$$

The proof is left to the reader.

6.55. *If $VA, VA' ; VB, VB' ; VC, VC' \dots$ are pairs of perpendicular lines through V , then*

$$V[A, A' ; B, B' ; C, C' ; \dots]$$

is an involution pencil.

Let any transversal meet the rays $VA, VA' ; VB, VB', \dots$ in $A, A' ; B, B' ; \dots$ respectively. Let VO be perpendicular to the transversal. Then, we have

$$OA.OA' = -OV^2 ;$$

Similarly, $OB.OB' = -OV^2,$

Hence $OA.OA' = OB.OB' = OC.OC' = \dots\dots\dots$

$\therefore AA' ; B,B' ; C,C'$ are point-pairs in involution.

$\therefore V [A,A' ; B,B' ; C,C' ; \dots\dots\dots]$ is an involution pencil.

Such an involution pencil is called *orthogonal*.

Note. The united points of the involution range determined by an orthogonal involution pencil on any transversal are evidently imaginary. Hence so must be the united rays of the orthogonal involution. These united rays of an orthogonal involution pencil are called the *circular lines* though the vertex of the pencil.

2A 6.6. **Theorem** In every involution pencil there exists just one pair of corresponding rays at right angles; and if more than one such pair exists then every pair of corresponding rays is at right angles.

Consider an involution in which the united rays VU_1, VU_2 are real. Let VA, VA' be the internal and external bisectors of the angle U_1VU_2 . Then

$$V(U_1U_2, AA') = -1.$$

Hence VA, VA' are a pair of corresponding rays of the involution of which VU_1, VU_2 are the united lines. Thus they belong to the given involution pencil. Moreover, they are at right angles.

In case, the united rays are imaginary, we take recourse to the Principle of Continuity and thus establish the result.

This proves the first part of the theorem.

Further, if two pairs of corresponding rays at right angles belong to an involution pencil then every pair of corresponding rays of the pencil must be at right angles, for an involution is completely determined by two pairs of elements and a system of line pairs at right angles drawn through a point determine an involution pencil.

✓ 6.71. Theorem. *An involution pencil projects into an involution pencil.*

Let $V [A.A' ; B.B' ; C.C' ; \dots\dots\dots]$
be a pencil in involution in a plane α .

Project to a plane π .

Let V' be the projection of V .

Let $VA, VA' ; VB, VB' \dots\dots\dots$ meet the axis of projection in $A.A' ; B.B' ; \dots\dots\dots$ so that VA projects into $V'A, VA'$ into $V'A$ etc.

Also, $(A.A' ; B.B' ; C.C' \dots\dots\dots)$ is a range in involution, being the range determined by the given involution pencil on the axis of projection.

Hence $V' (A.A' ; B.B' ; C.C' ; \dots\dots\dots)$ is an involution pencil.

Cor. Since it is always possible to project two angles in the α plane into angles of given magnitudes on to the π plane, an involution pencil can be so projected that the angles between two pairs of corresponding rays project into right angles. Since two pairs of corresponding rays are at right angles in the projected pencil in involution in the π plane, every pair of corresponding rays is at right angles i.e., the involution pencil in the π plane is orthogonal.

Hence an involution pencil can always be projected into an orthogonal involution pencil.

✓ 6.8. Involution Properties of a complete quadrangle and a complete quadrilateral.

✓ 6.81. Theorem *Any transversal cuts the pairs of opposite sides of a complete quadrangle in pairs of points belonging to an involution.*

Let $ABCD$ be the quadrangle and let any trans-

versal t cut its pairs of opposite sides in the pairs of points P, P_1 ; Q, Q_1 ; R, R_1 as shown in the figure

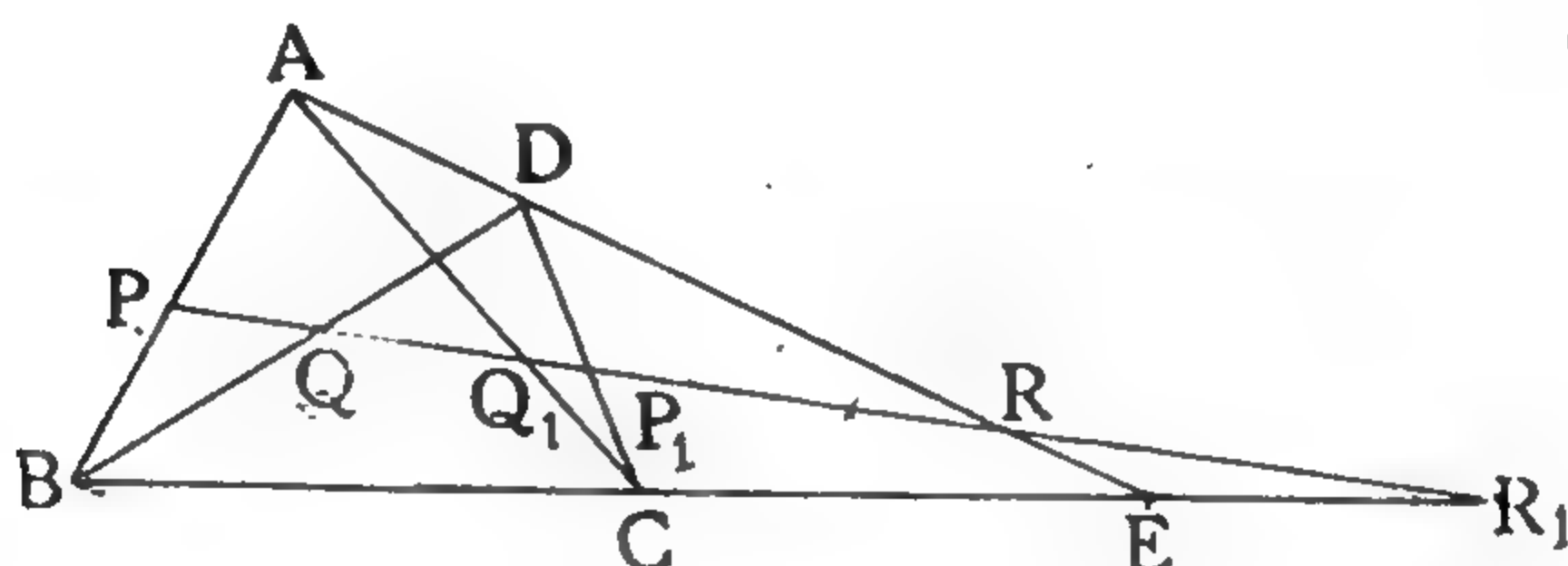


Fig. 53.

Let AD, BC meet in E . We have

$$\begin{aligned} (PQRR_1) &= B(PQRR_1) \\ &= (ADRE), \text{ on the transversal } AD \\ &= C(ADRE) \\ &= (Q_1P_1RR_1), \text{ on the transversal } t \\ &= (P_1Q_1R_1R) \end{aligned}$$

Hence P, P_1 ; Q, Q_1 ; R, R_1 are in involution.

Cor. If a transversal is drawn through a diagonal point, this diagonal point will be a double point of the involution formed on the transversal.

6.82. Theorem. The lines joining any point to the pairs of opposite vertices of a complete quadrilateral form an involution pencil.

Let the pairs of opposite vertices be A, A_1 ; B, B_1 ; C, C_1 and let T be any given point. Let TC_1 cut AB in X and A_1B_1 in Y .

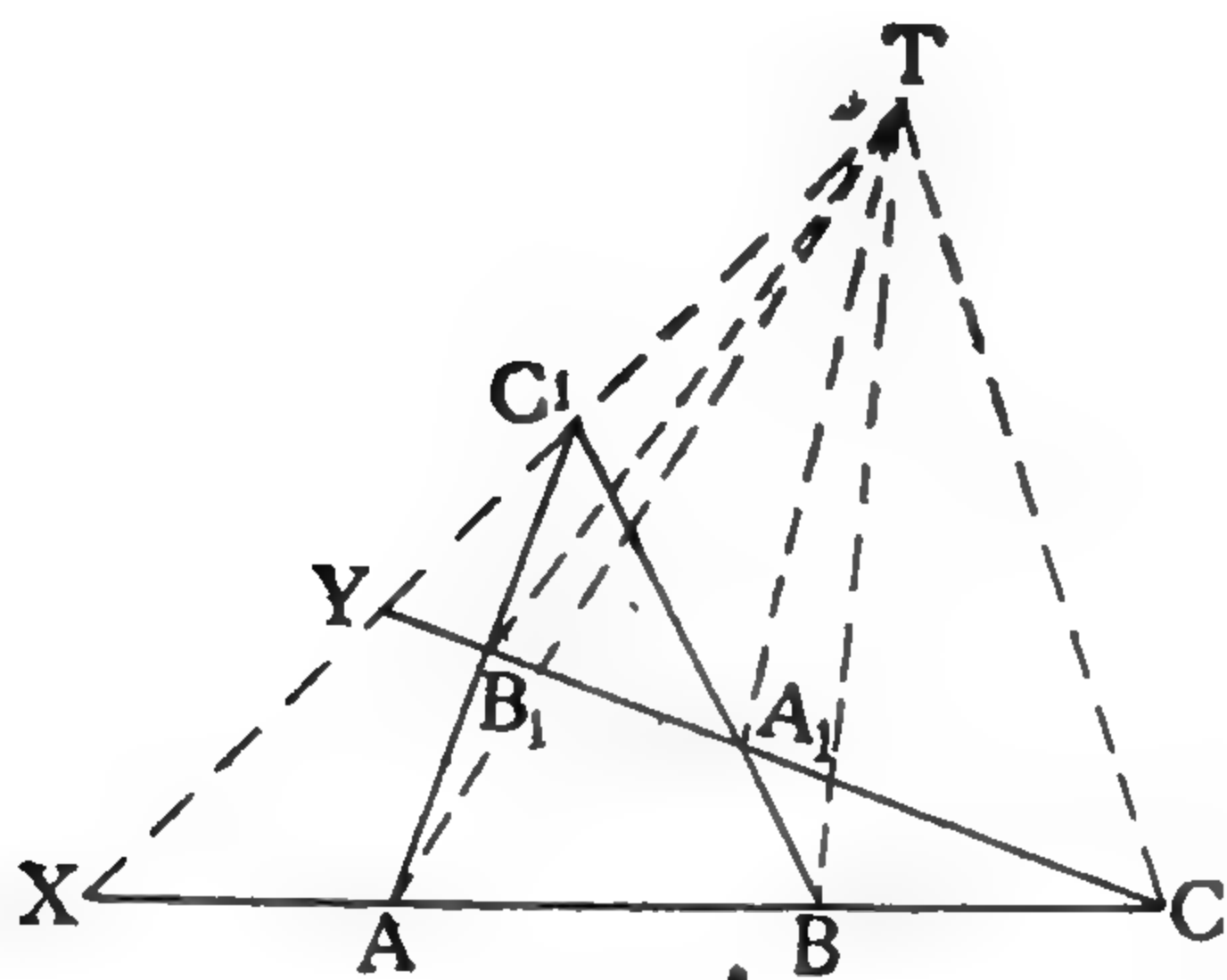


Fig. 45.

We have

$$\begin{aligned}
 T(ABCC_1) &= (ABCX), \text{ on the transversal } AB \\
 &= C_1(ABCX) \\
 &= (B_1A_1CY), \text{ on the transversal } A_1B_1 \\
 &= (A_1B_1YC) \\
 &= T(A_1B_1YC) = T(A_1B_1C_1C)
 \end{aligned}$$

Thus $TA, TA_1; TB, TB_1; TC, TC_1$ belong to the same involution.

Cor. If T is taken on a diagonal line of a quadrilateral, this diagonal line is a double line of the involution pencil formed at the point.

6.9. To show 'ab' initio that a system of point pairs are in involution. A system of point pairs has been defined to be in involution, if a rational, algebraic, one to one correspondence exists between the elements of each pair. Also, a necessary and sufficient condition for point pairs to be in involution has been obtained in terms of an equality involving cross-ratios so that in order to establish that any given system of point pairs are in involution, we have generally shown that this equality is true. It is, however, sometimes possible to establish this fact 'ab' initio without employing the notion of cross ratios.

Let a system of point-pairs lying on the same base be given by means of series of steps of construction.

If starting from any point P of the base and performing the given series of steps of construction, we arrive at the only point P' and, if starting from P' and performing the same steps, we arrive at P , then the correspondence is reversible and one to one. To determine whether the correspondence is rational algebraic, we have to take note of the analytical equivalents of the given steps of construction.

Thus an examination along the lines suggested above of the nature of the steps of construction which determine any given system of point-pairs on the same base, may enable us to determine directly if the point pairs are in involution.

By an illustration we consider the following result already proved in § 6.21 P. 86.

If a *variable* pair of points P, P' is harmonically conjugate *w. r. t.* a fixed pair of points U_1, U_2 , then P, P' determine an involution.

Since for a given position of P , there is only one position of P' , *viz.*, the fourth harmonic of P for U_1, U_2 and *vice versa*, therefore, P, P' are in (1, 1) correspondence.

Also since $(U_1U_2, PP') = -1 = (U_1U_2, P'P)$ the correspondence is reversible.

Since the fourth harmonic P' of P can be determined by means of a series of steps of construction involving the use of straight lines only (Ref. § 5.8, P. 78), and the equation of every straight line is rational, algebraic and of the first degree in the variable co-ordinates, we see that the correspondence is rational, algebraic.

Hence, P, P' determine an involution.

Evidently, U_1, U_2 are the united points of the involution.

Exercises

1. Prove that any involution range can be projected into a system of point-pairs having a common mid-point.

2. $(A, A'; B, B'; C, C')$ are three point-pairs in involution; show that

$$BC = \frac{BA \times CA'}{B'A'} + \frac{AC \times A'B}{A'C'}$$

3. P, P' is any variable point-pair of an involution determined by the double points U_1, U_2 : show that $(PU_1 \times P'U_2) / PP'$ is constant.

4. If U_1, U_2 be the double points of the involution determined by A, A' and B, B' show that

$$(A, B; A'B'; U_1, U_2) \text{ and } (A, B'; A'B; U_1, U_2)$$

are two involutions.

5. Two triangles ABC and $A'B'C'$ are in plane perspective. $BC, B'C'$ intersect in P ; $CA, C'A'$ in Q and $AB, A'B'$ in R . AA', BB', CC' meet the line PQR in P', Q', R' . Show that $(P, P'; Q, Q'; R, R')$ is an involution.

6. Through a point V in the plane of a triangle ABC , lines $V\alpha, V\beta, V\gamma$ are drawn parallel to BC, CA, AB ; show that

$$V(A, \alpha; B, \beta; C, \gamma)$$

is an involution pencil.

7. If straight lines are drawn through any point parallel to the opposite pairs of sides of a quadrangle, they form an involution pencil.

8. $A, A'; B, B'; C, C'; D, D'$ are fixed point-pairs of an involution, H is a variable point on the base; P, Q, R, S are the harmonic conjugates of H with respect to the four-point pairs. Show that $(PQRS)$ is constant.

9. If a transversal cut the sides of a triangle ABC in X, Y, Z and O is any point in the plane of the triangle, then $O(A, X; B, Y; C, Z)$ is an involution.

10. If a transversal cuts the sides of a triangle ABC in X, Y, Z and points X', Y', Z' are taken on the transversal so that $(X, X'; Y, Y'; Z, Z')$ is an involution, then AX', BY', CZ' are concurrent.

11. A transversal cuts the three sides BC, CA, AB of a triangle in P, Q, R ; and also cuts three concurrent lines through A, B and C respectively in P', Q', R' . Prove that

$$PQ' \cdot QR' \cdot RP' = -P'Q \cdot Q'R \cdot R'P.$$

12. $A, A'; B, B'; C, C'; D, D'$ are four point-pairs in involution and P, Q, R, S are the mid-points AA', BB', CC', DD' ; show that

$$(PQRS) = (ABCD) = (A'B'CD').$$

13. Construct the sixth point C' of an involution of which the five points $A, A'; B, B'; C$ are given.

CHAPTER VII

ELEMENTARY HARMONIC PROPERTIES OF THE CIRCLE AND THE CONIC

7.1. Inverse points. Def. Pairs of points on a diameter of a circle which are harmonically conjugate w. r. t. its extremities, are called inverse points of the circle.

Each element of the pair is said to be *inverse* of the other. From the definition it follows that the totality of pairs of points inverse w. r. t. a circle and lying on a diameter are in involution, the extremities of the diameter being the united points of the involution.

Ex. If P is the inverse of Q w. r. t. a circle, centre C , then $CP \cdot CQ = (\text{radius of the circle})^2$.

Polar and Pole. Def. If a variable line is drawn through a fixed point P meeting a given circle in Q, S and if R be the harmonic conjugate of P w. r. t. Q, S ; then the locus of R is called the Polar of P w. r. t. the circle and P is called the pole of the locus of Q .

Theorem. The polar of a point w. r. t. a circle is a straight line.

Let P be a given point and PQS any chord through P meeting the given circle in Q, S . Let R be the fourth harmonic of P w. r. t. Q, S . We are to show that the locus of R is a straight line.

Let N be the inverse of P .

Let M be the middle point of QS .

$$(PR, QS) = -1,$$

$$\frac{2}{PR} = \frac{1}{PQ} + \frac{1}{PS}.$$

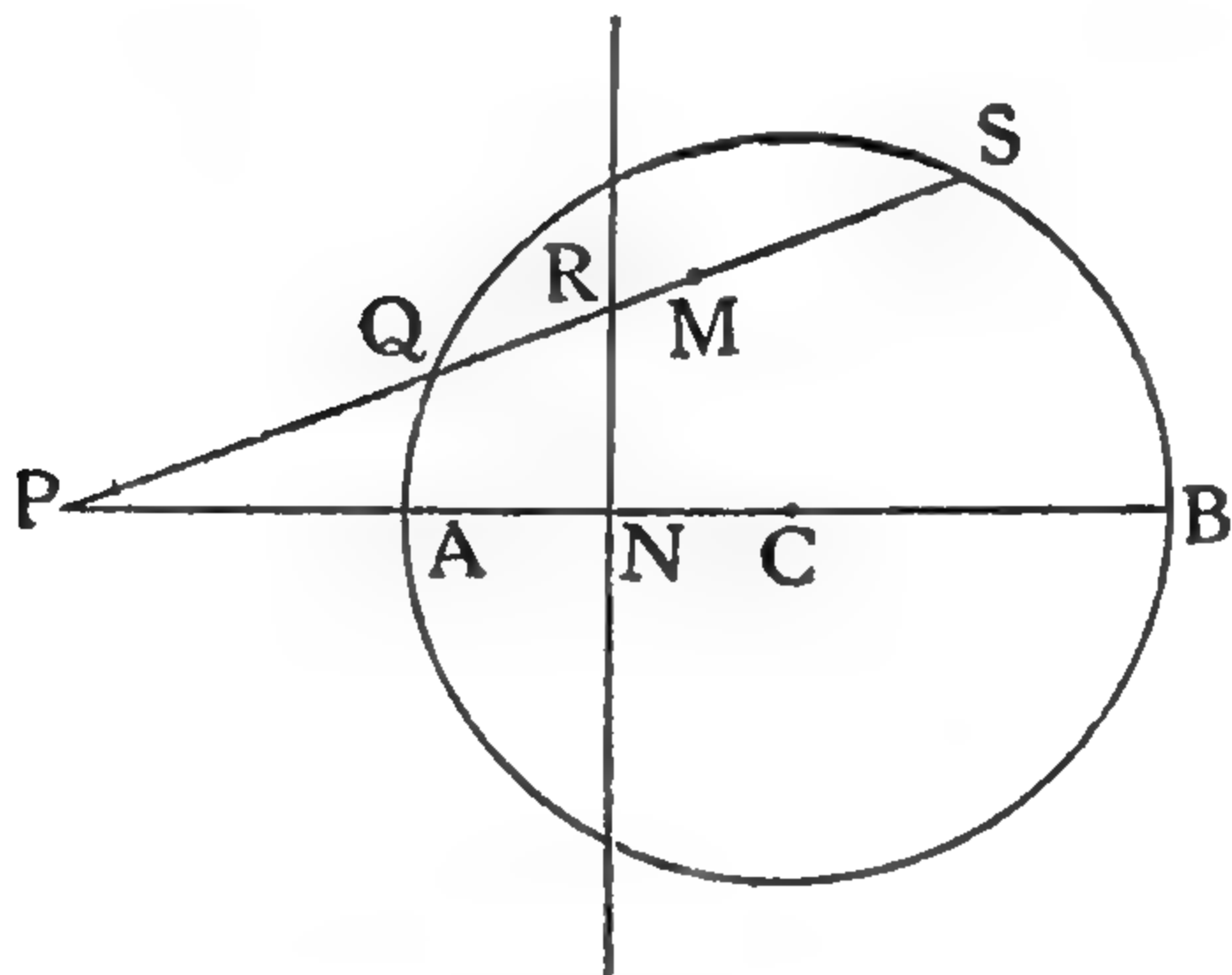


Fig. 55.

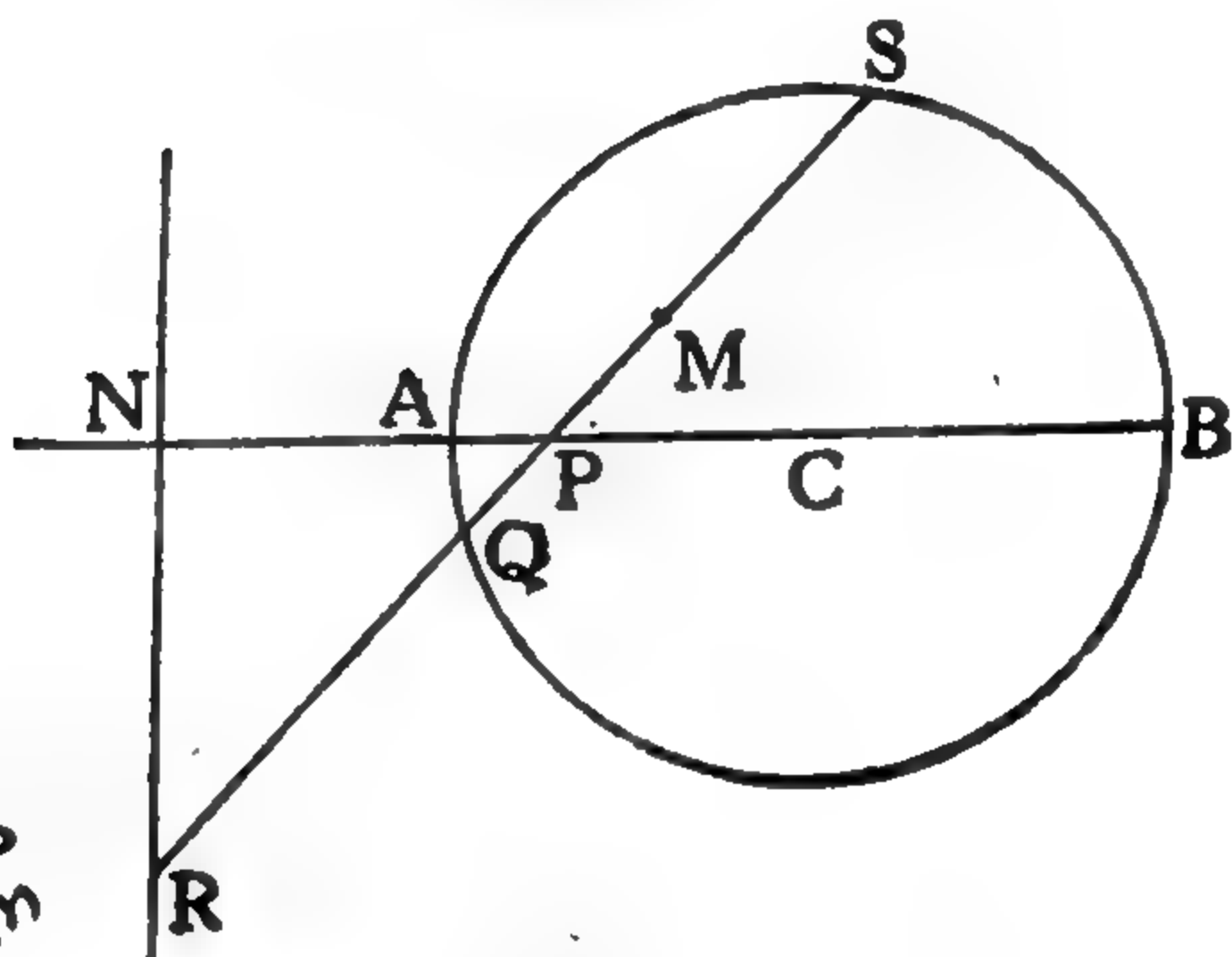


Fig. 56.

$$= \frac{PQ + PS}{PQ \cdot PS} = \frac{2PM}{P \cdot QPS}$$

M being the middle point of QS.

$$\therefore PQ \cdot PS = PM \cdot PR \quad \dots \dots \dots (1).$$

Again, $\because (PN, AB) = -1$,

$$\begin{aligned} \therefore \frac{2}{PN} &= \frac{1}{PA} + \frac{1}{PB} \\ &= \frac{PA + PB}{PA \cdot PB} \end{aligned}$$

$$\begin{aligned} PQ + PS &= 2PM \\ 2PM &= PQ + PS \\ 2PM &= PQ + PS \end{aligned}$$

$$= \frac{2PC}{PA \cdot PB},$$

C being the mid-point of AB .

$$\therefore PA \cdot PB = PC \cdot PN.$$

$$\text{But } PQ \cdot PS = PA \cdot PB.$$

$$\therefore PQ \cdot PS = PC \cdot PN \dots \dots \dots (2)$$

Therefore from (1) and (2).

$$PC \cdot PN = PM \cdot PR.$$

$\therefore RNCM$ is a cyclic quadrilateral.

$\therefore \angle CNR = \angle CMR = 1\text{rt. angle}$; M being the middle point of QS , $CM \perp QS$.

Hence the locus of R is a fixed straight line *viz.*, the line through the inverse of P perpendicular to the diameter through P . Thus the polar of P is a straight line.

Ex. If PQS , $PQ'S'$ are two chords of a circle through P , the polar of P passes through the other two diagonal points of the quadrangle $QQ'S'S$.

Note 1. The point N which is the inverse of P lies inside or outside the circle according as P lies outside or inside the circle.

Thus the polar of a point P cuts the circle in real or imaginary points according as P lies outside or inside the circle.

Let the polar of a point P outside the circle meet it in T and T' . The line PT meets the circle in two points Q , S ; Q being the same as T . Since $(PT, QS) = -1$, and two points Q and T are coincident, therefore S must also coincide with them. (§5.41 P. 71). Thus PT meets the circle in two coincident points and is, therefore, a tangent. Similarly PT' is a tangent. Hence the polar of a point P outside a circle is the chord TT' joining the points of contact of the tangents drawn from P .

The inverse of a point P lying on a circle is the point P itself, so that the polar of P is the line through P perpendicular to the diameter through P *viz.*, the tangent at P . Hence the polar of a point on the circle is the tangent to the circle at the point; in other words, the pole of a tangent to a circle is its point of contact.

2. The polar of any point P will be denoted by the corresponding small letter p and the pole of any line p will be denoted by the corresponding capital letter P .

✓ 7.2. Theorem. *The polar of the centre of a circle is the line at infinity in the plane.*

Since every chord through the centre C is bisected at C , the fourth harmonic of C w.r.t. the points of intersection is at infinity. Hence the polar of the centre is the line at infinity.

The same result can also be stated as follows :—

The pole of the line at infinity w.r.t. a circle is the centre of the circle.

gm 7.3 Salmon's Theorem. *If PA , QB be drawn perpendicular to the polars of Q and P respectively, w.r.t. a circle centre O , then*

$$\frac{PA}{QB} = \frac{OP}{OQ}.$$

Let N , M be the inverse points of P and Q respectively.

Draw PK , QL perpendicular to OQ , OP respectively.

Since $\angle K = \angle L = \text{rt. } \angle$, therefore $PKLQ$ is a cyclic quadrilateral.

$$\therefore OP \cdot OL = OQ \cdot OK$$

$$\text{or } \frac{OP}{OQ} = \frac{OK}{OL} \dots\dots(1)$$

Again, since N is the inverse of P , therefore

$$ON \cdot OP = r^2, \text{ } r \text{ being the radius of the circle.}$$

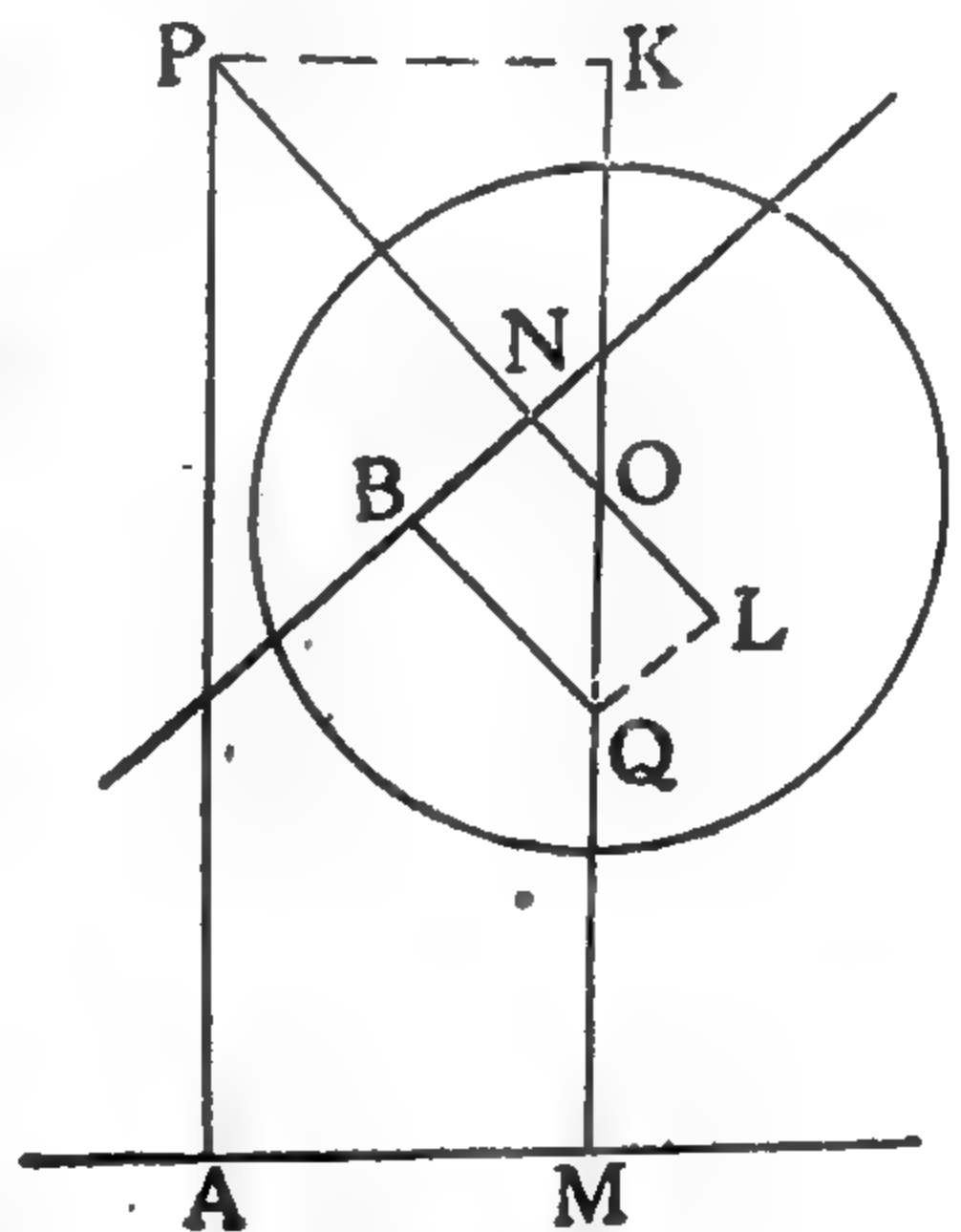


Fig. 57.

Similarly $OM.OQ=r^2$.

$$\therefore ON.OP = OM.OQ.$$

$$\text{or } \frac{OP}{OQ} = \frac{OM}{ON} \quad \dots \dots \dots (2)$$

From (1) and (2),

$$\begin{aligned} \frac{OP}{OQ} &= \frac{OM}{ON} = \frac{OK}{OL} = \frac{OM-OK}{ON-OL} \\ &= \frac{KM}{LN} = \frac{PA}{QB} \end{aligned}$$

✓ 7.4. Theorem. If the polar of P passes through Q , then the polar of Q passes through P .

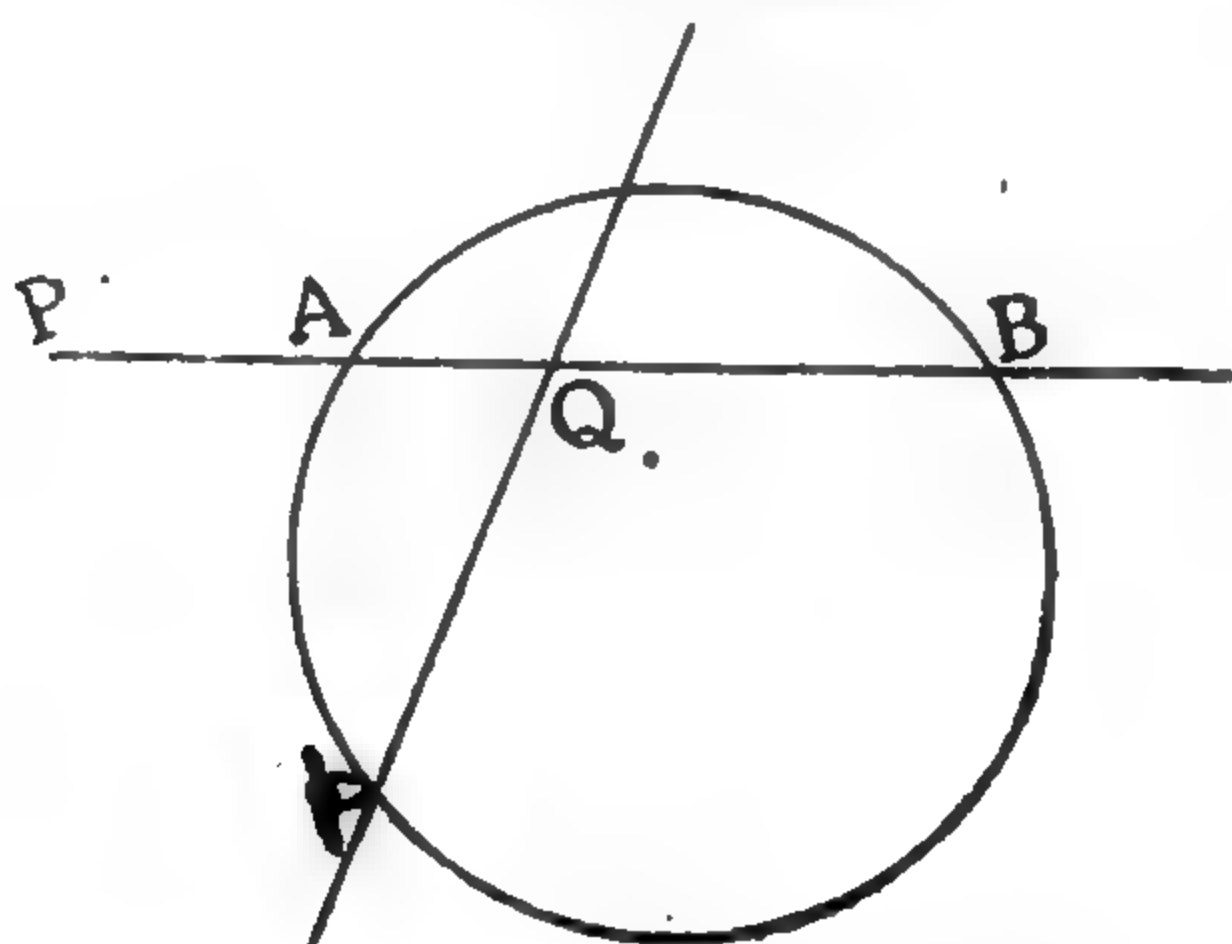


Fig. 58.

Join PQ , and let it meet the circle in A, B .

Q , being a point on the polar of P , we have

$$(PQ, AB) = -1,$$

so that P is the fourth harmonic of Q w.r.t. A and B .

Therefore the polar of Q passes through P .

Note. If P and Q are both outside the circle, PQ does not meet the circle in real points. The result is however to be seen to be true in that case also, from the principle of continuity.

Cor. If a number of points are collinear, their polars w.r.t. a circle are concurrent. See Ex 2 Page 110.

The point of concurrence is the pole of the line on which the points lie.

Def. A range of points and the corresponding pencil of their polars are said to be *reciprocal*.

Def. Two points are said to be *conjugate w.r.t. a circle* if each lies on the polar of the other.

A pair of points is conjugate *w.r.t.* a circle if and only if it is harmonically conjugate *w.r.t.* the pair of points in which the line joining the given pair meets the circle.

Ex. Show that there exist infinite pairs of conjugate points on a given line.

7.41. Theorem. *Pairs of points conjugate w.r.t. a circle which lie on a line determine an involution of which the double points are the points of intersection of the line with the circle.*

This result follows from the fact that every such pair is harmonically conjugate *w.r.t.* the points in which the line cuts the circle.

7.42. Theorem. *If the polars of P and Q intersect in R, then PQ is the polar of R.*

Since the polar of P passes through R , therefore the polar of R passes through P .

Similarly, the polar of R passes through Q .

Therefore PQ is the polar of R .

The theorem can also be differently worded thus :—

The meet of the polars of two points is the pole of their join. ΔPQR is called the triangle of self-correspondence. Δ

7.43. Theorem. *If the pole of p lies on q , the pole of q lies on p .*

This follows from § 7.4.

Cor. If a number of lines are concurrent, their poles w.r.t. a circle are collinear. See Ex. Page 110.

The line of collinearity is the polar of the vertex of the given pencil of lines.

Def. A pencil of lines and the corresponding range of its poles are said to be *reciprocal*.

Def. Two lines are said to be *conjugate w.r.t.* a circle if each contains the pole of the other.

7.44. Theorem. *If the poles of p and q lie on r , then (pq) is the pole of r .*

Since the pole of p lies on r , therefore the pole of r lies on p .

Again, since the pole of q lies on r , therefore the pole of r lies on q .

Hence pole of r is (pq) .

The theorem can also be differently worded thus :—

The join of the poles of two lines is the polar of their meet.

(7.45) Theorem. *Pairs of lines through a point conjugate w.r.t. a circle are in involution, the united rays of the pencil being the tangents from the point to the circle.*

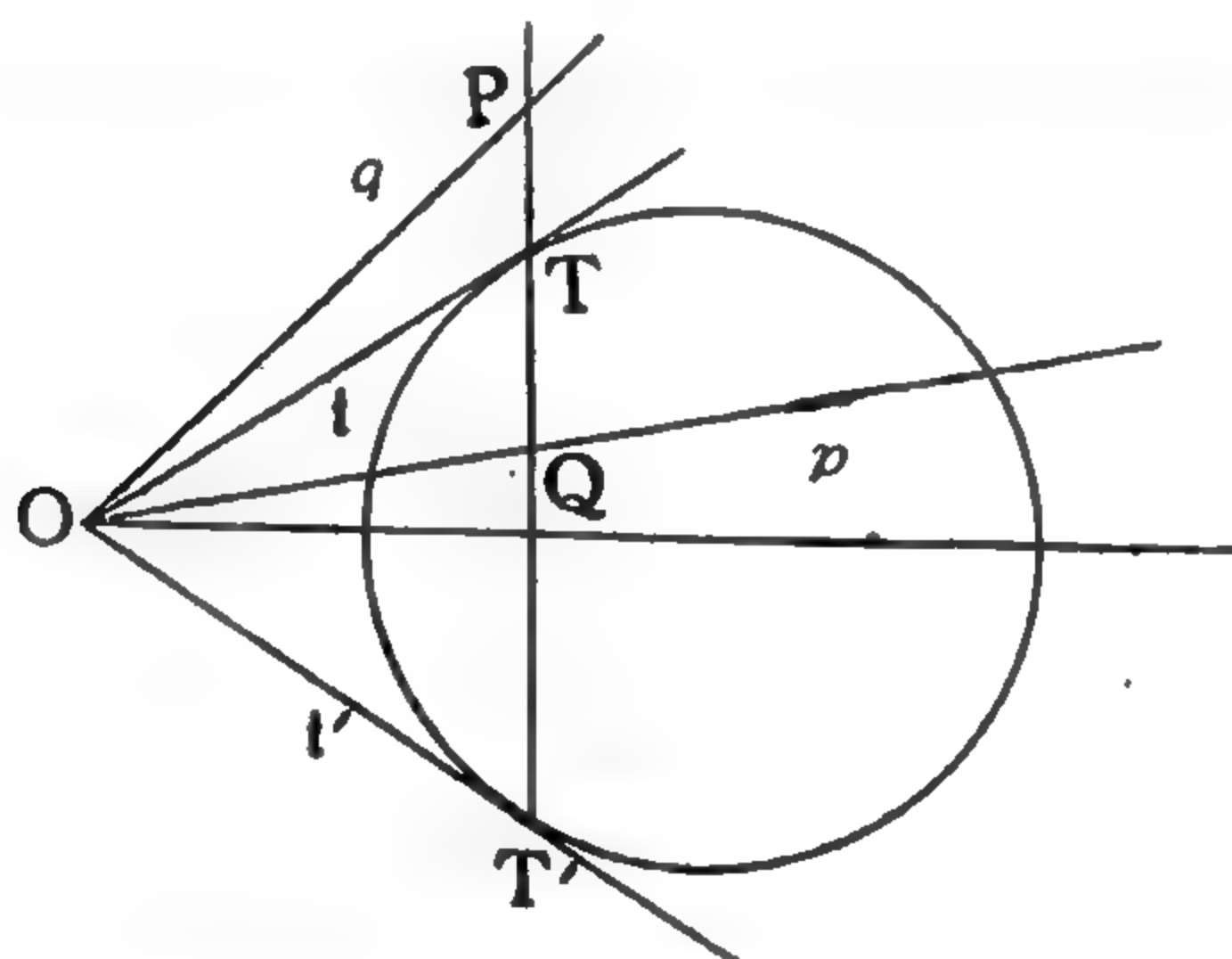


Fig. 59

Let O be the given point and p and q a pair of lines through O conjugate w.r.t. a given circle.

Let P and Q be the poles of p and q for the circle so that they lie on q and p respectively.

Then, by the above theorem, PQ is the polar of O , the meet of p and q .

Again, if PQ meets the circle in T, T' , OT and OT' are tangents to the circle.

Also, since P, Q is a pair of conjugate points for the circle, therefore

$$i(TT', PQ) = -1$$

$$\text{or } O(TT', PQ) = -1 \text{ i.e. } O(tt', pq) = -1.$$

Thus every pair of conjugate lines for the circle through O is harmonic w.r.t. the pair of tangents to the circle from O . Hence these lines belong to an involution pencil of which the tangents from the vertex to the circle are the united lines.

Note. When O lies inside the circle, OT, OT' are imaginary but the truth of the theorem follows in this case from the Principle of Continuity.

7.5. Conjugate diameters. A pair of conjugate lines through the centre of a circle is called a pair of conjugate diameters.

Theorem. A pair of conjugate diameters of a circle are at right angles.

Let AA' be any diameter of the circle. Then the diameter conjugate to AA' is the join of the centre O to the pole of AA' . But the pole of AA' is a point at infinity, say A_1 .

Again, since the line joining a point to the centre is perpendicular to the polar of the point therefore OA_1 is perpendicular to AA' . Hence the theorem.

The result of § 7.45 now leads us to the following important theorem:—"Pairs of conjugate diameters of a circle determine an orthogonal involution pencil at the centre, the united rays of the pencil being the imaginary pair of tangents from the centre to the circle."

7.51. Circular points. Circular lines. Let p, p' be any pair of corresponding rays of the orthogonal involution pencil at the centre of a circle. They will

meet the line at infinity in a pair of points P, P' which determine an involution, the united points W_1, W_2 being the imaginary points of intersection of the circle and the line at infinity.

Since every line through P is parallel to p and every line through P' is parallel to p' the point P, P' will subtend a right angle at every point in the plane. Hence, if O_1 is any other point in the plane, then $O_1 P$ and $O_1 P'$ determine orthogonal involution at O_1 , the united rays of the pencil at O_1 being the join of O_1 to W_1, W_2 . Thus $O_1 W_1, O_1 W_2$ are the united lines of the orthogonal involution pencil of conjugate diameters of any circle having its centre at O_1 . Therefore, every circle cuts the line at infinity in the same two points W_1, W_2 which are called the **circular points**.

The lines joining any point to the circular points are called the **circular lines** through the point.

Obviously, the circular lines through a point are tangents to every circle having its centre at the point. Consequently, concentric circles may be said to touch each other at the circular points.

Ex. 1. Any two points on the line at infinity which subtend a right angle at the centre of a circle are conjugate w. r. t., the circle.

7.52. Locus of the middle points of a system of parallel chords.

Let M be the middle point of any chord PQ of a

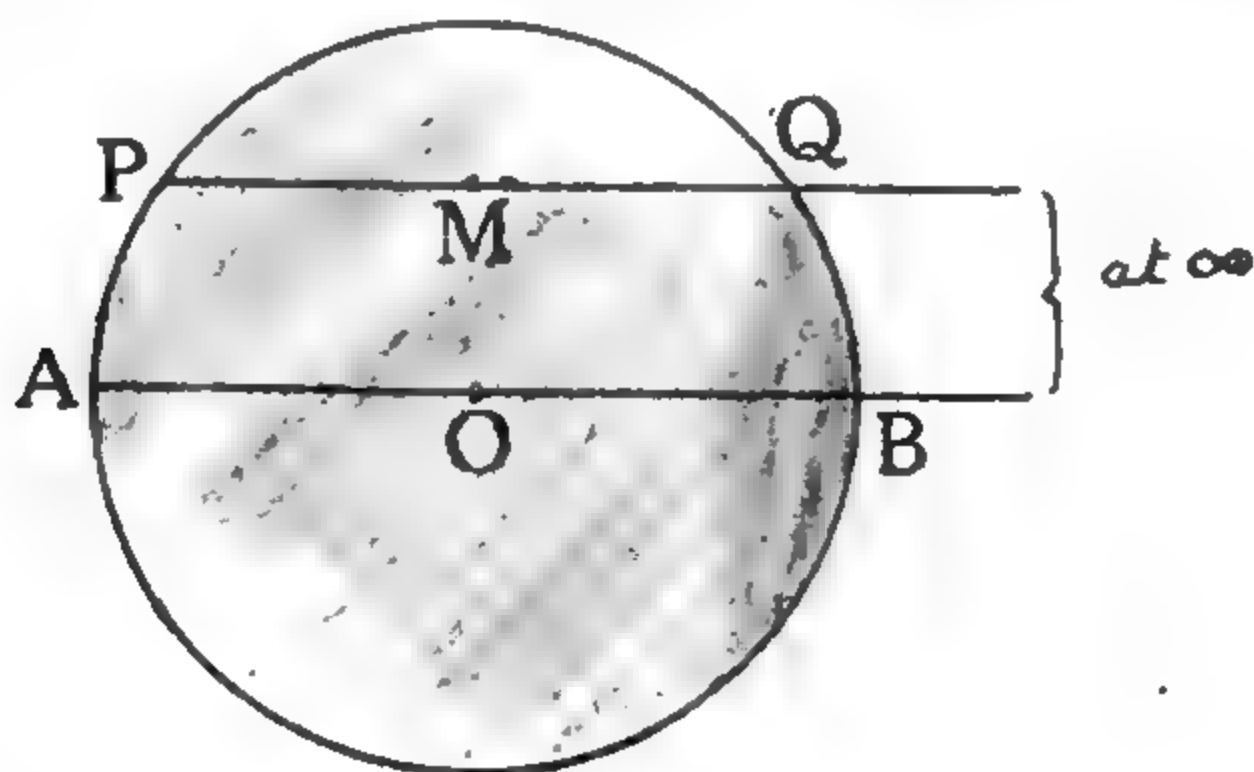


Fig. 60.

system of parallel chords.

Let AOB be the diameter of the circle parallel to PQ .

Let W be the point at infinity common to AB and PQ .

We have

$$(PQ, MW) = -1 \text{ and } (AB, O, W) = -1,$$

i.e., M and O are the fourth harmonics of W for P, Q and A, B respectively. Thus MO is the polar of W .

Hence the locus of M is a diameter conjugate to AB and is therefore perpendicular to AB .

Thus the locus of the mid-points of a system of parallel chords of a circle is a diameter perpendicular to the chords.

Cor. 1. Of two conjugate diameters, each bisects chords parallel to the other.

§ 6. Orthogonal circles. Def. Two circles are said to be orthogonal if the angle between the tangents to the circles at their point of intersection is a right angle.

Let PT, PT' be the tangents to two circles which

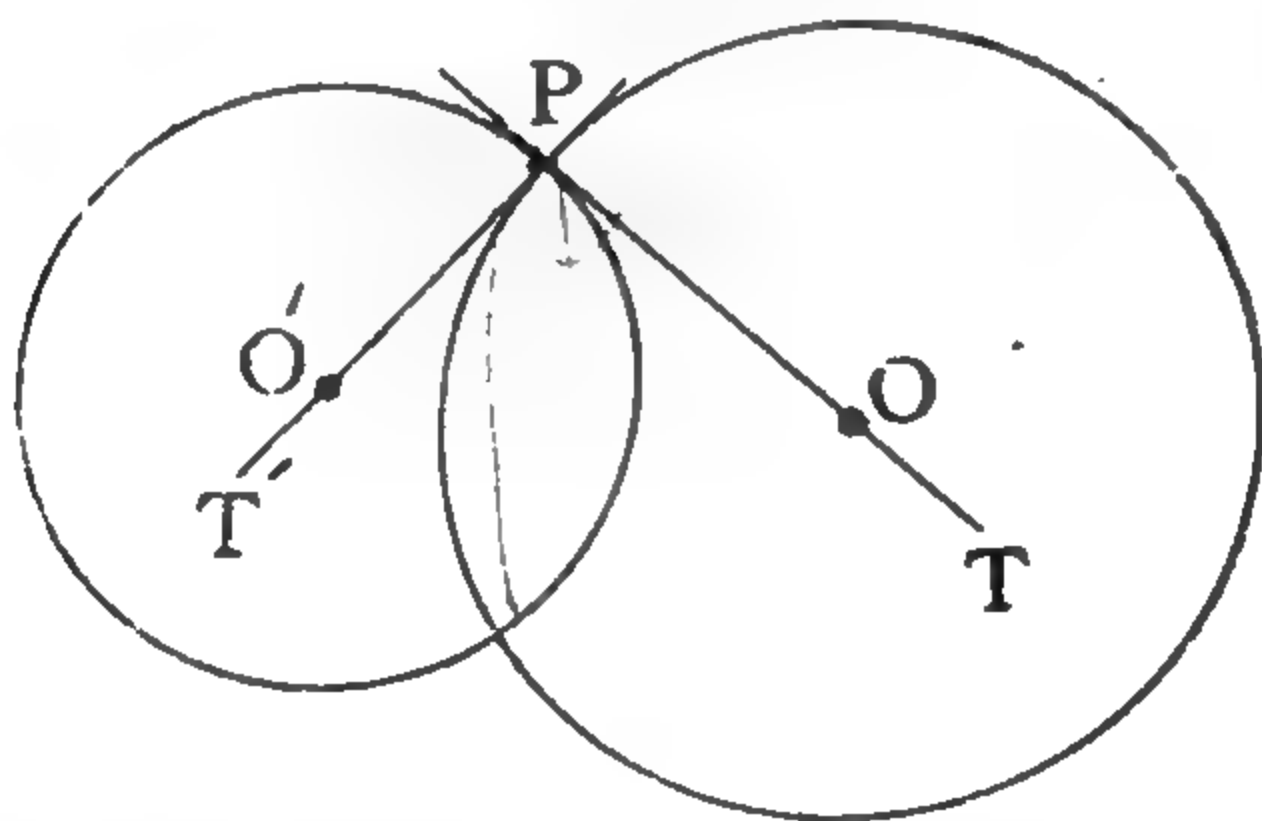


Fig. 61.

cut each other orthogonally at P . Then $\angle TPT' = 1 \text{ rt } \angle$.

Also, if OP be the radius of the circle to which PT' is a tangent,

$$\angle O'PT = 1 \text{ rt. } \angle.$$

$$\therefore \angle O'PT = \angle T'PT.$$

Hence PO coincides with PT .

Similarly, PO' coincides with PT' .

Hence if two circles are orthogonal, the radius of each through a point of intersection is a tangent to the other.

Def. The power of a point *w. r. t.* a circle is the square of the length of the tangent from the point to the circle.

Hence if two circles are orthogonal, the power of the centre of each with respect to the other equals the square of its radius.

Ex. 1. If the radii of two circles through a common point contain a right angle, the circles are orthogonal.

Ex. 2. If the power of the centre of a circle *w. r. t.* another circle is equal to the square of the radius of the former, the circles are orthogonal.

Ex. 3. If two circles with centre O and O' intersect in P , show that the necessary and sufficient condition that the circles be orthogonal is $OP^2 + O'P^2 = OO'^2$.

✓ **7.61. Theorem.** If P and Q are conjugate points *w. r. t.* a circle S , the circle on PQ as diameter cuts S orthogonally.

Let PQ meet S in A and B .

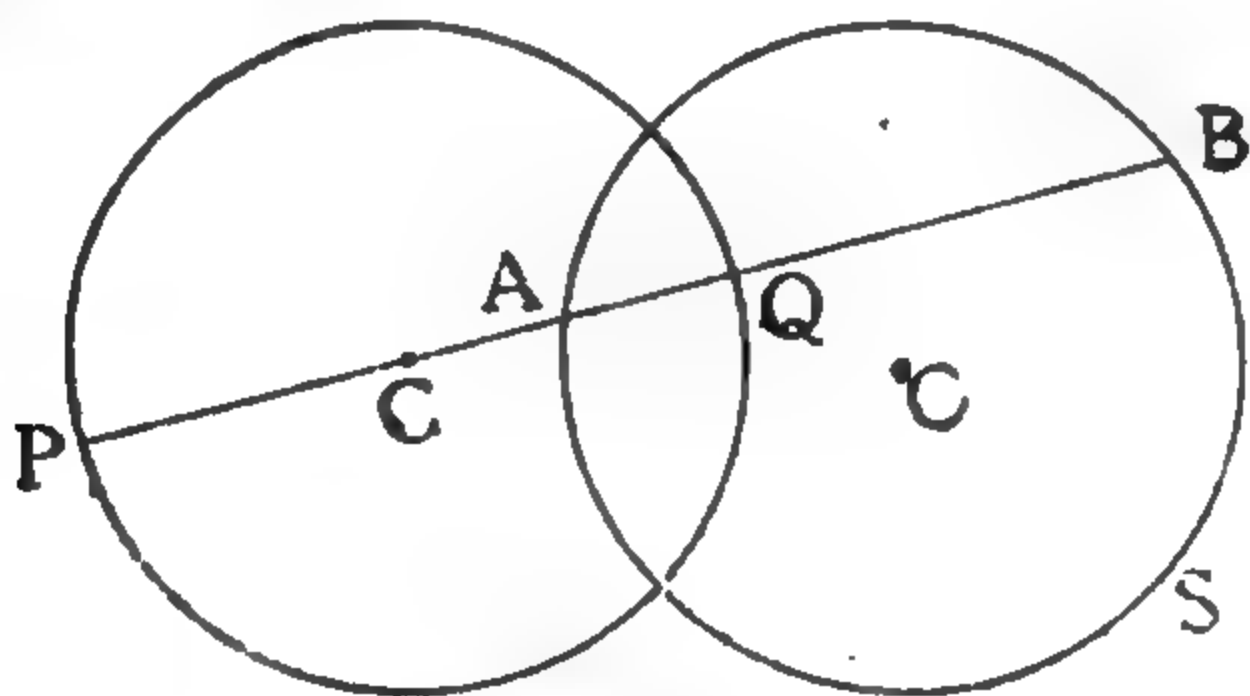


Fig. 62.

Since P, Q are conjugate points w. r. t. S , we have

$$(PQ, AB) = -1,$$

$$\therefore CA \cdot CB = CP^2$$

i.e., the power of C , w. r. t. S = square of its radius.

Hence the two circles are orthogonal.

✓ **Cor.** If $(PQ, AB) = -1$, the circle on PQ (AB) as diameter will cut every circle through A and B (P and Q) orthogonally.

Ex. Show that, if AB is a diameter of a circle which cuts two given circles orthogonally, the polars of A with respect to the two circles intersect in B .

7.7. Conjugate triangles. Let P, Q, R be the vertices of a triangle. Let p, q, r , be the polars of P, Q, R respectively w. r. t. a given circle, centre O . Then the two triangles PQR, pqr are said to be *conjugate triangles* w. r. t. the given circle.

Conjugate triangles are also said to be *Reciprocal*.

Ex. Show that PQ, QR, RP are the polars of $(pq), (qr), (rp)$ respectively.

Def. Self-conjugate triangles. A triangle which coincides with its own conjugate is said to be *self-conjugate*. It is also called a *self-polar* or *self-reciprocal* triangle. Evidently in such a triangle, each side is the polar of the opposite vertex and each vertex is the pole of the opposite side.

One of the vertices P of such a triangle can be chosen arbitrarily. The other vertex, then, is restricted to be on the polar of P and can be taken anywhere on it. Then the third vertex R is fixed, being the point of intersection of the polars of P and Q .

If P lies inside the circle, the polar p of P does not meet the circle in real points so that the remaining two vertices Q and R (which lie on p) are outside the circle.

If P lies outside the circle, the polar of P will cut the circle in real points. The vertex Q , now, may be chosen

inside or outside the circle. The third vertex R will, lie outside or inside the circle according as Q lies inside or outside.

Thus two vertices of a self-conjugate triangle lie outside and one inside the circle.

Theorem. The ortho-centre of a triangle self-conjugate for a circle lies at the centre.

Let PQR be a triangle self-conjugate for a circle, centre O .

Since QR is the polar of P , OP is perpendicular to QR .

Similarly, OQ is perpendicular to RP and OR is perpendicular to PQ .

Hence O is the ortho-centre of the triangle PQR .

Cor. Let OP meet QR in L and the circle in A and B . Then $(PL, AB) = -1$.

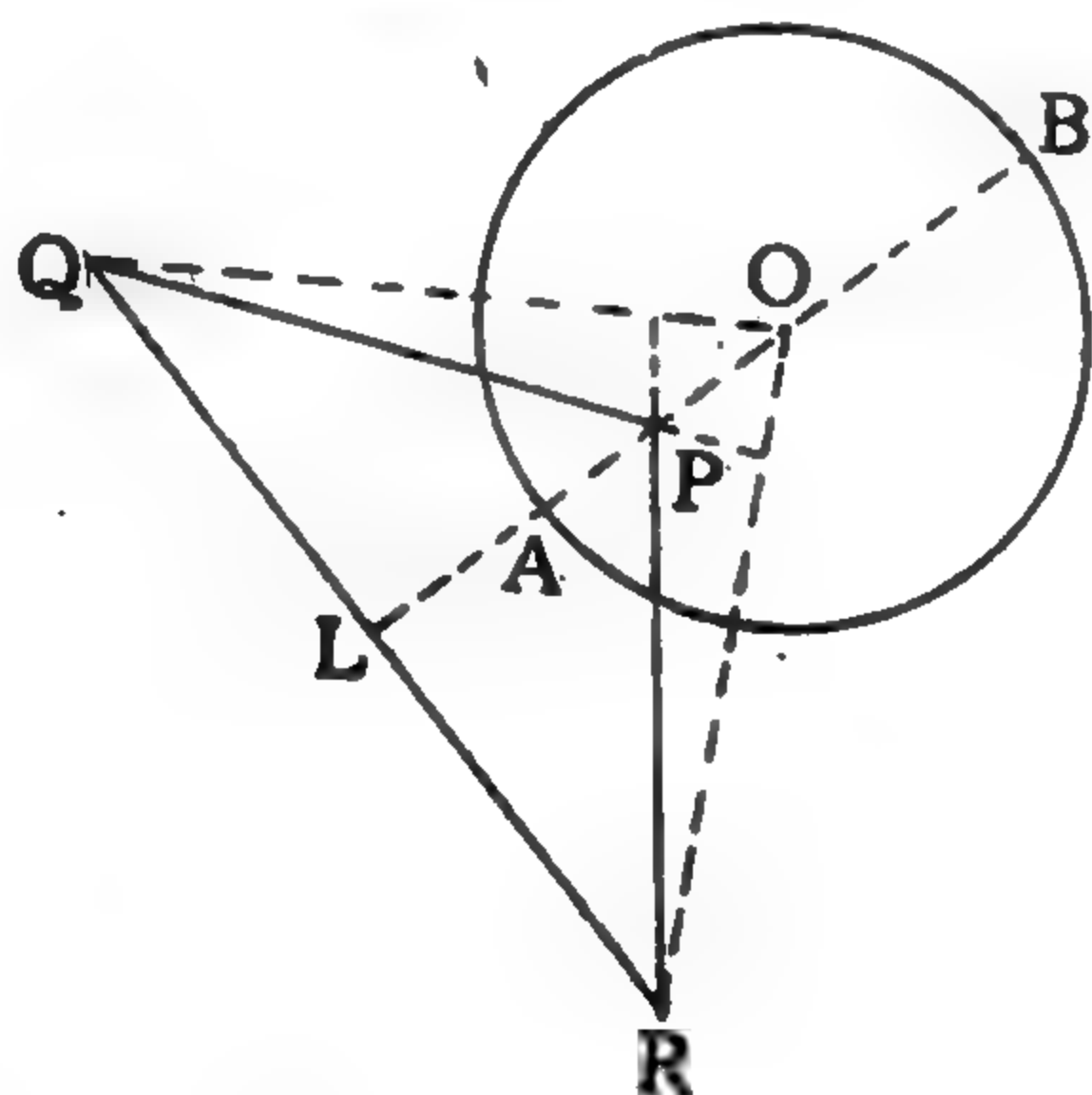


Fig. 63.

$\therefore OP \cdot OL = OA^2 = r^2$, r being the radius of the circle.

Thus, there is only one circle for which a given triangle PQR is self-conjugate, the centre lying at the

ortho-centre O of the triangle and the (radius)² being equal to $OP \cdot OL$, where L is the point of intersection of OP and QR .

Moreover, r^2 is positive and consequently r real only when OP and OL are both of the same sense i. e., when O lies outside the triangle PQR , which is possible only if the triangle is obtuse-angled. Hence we have the

Theorem. *Given a triangle, there is only one circle for which the triangle is self-conjugate and this circle is real only if the triangle is obtuse-angled.*

7.72. The circle is called the **polar circle** of the triangle.

Ex. 1. Any three points A, B, C are taken on the sides QR, RP, PQ of a triangle PQR which is given to be self-conjugate for a given circle. Show that the circles on PA, QB, RC as diameters cut the given circle orthogonally.

Since PQR is self-conjugate w. r. t. the circle, therefore QR is the polar of P .

If A is any point on QR , A and P are conjugate points for the circle.

Therefore the circle on AP as diameter will cut the given circle orthogonally.

Similarly, for the circles on BQ, CR as diameters.

Ex. 2. The circles drawn on the sides of a triangle as diameters cut the polar circle of the triangle orthogonally.

Exercises

1. A variable chord PQ of a circle passes through a fixed point, show that the tangents at P, Q meet on a fixed line.

2. PT, PT' are the tangents to a circle from a variable P on a fixed line, prove that TT' passes through a fixed point.

3. If A and B be conjugate points of a circle and P , the mid-point of AB , then the tangents from P to the circle are of length PA .

4. The vertex P of a triangle PQR which is self-conjugate w. r. t. a circle lies inside the circle; show that the chord of the circle bisected at P is parallel to QR .

HARMONIC PROPERTIES OF CIRCLES 111

5. T is the pole of a chord AB of a circle ; the tangent at a point P of a circle meets TA , TB , AB at Q , R , S respectively, show that TP is the polar of S and $(PS, QR) = -1$.

6. T is the pole of a chord AB of a circle ; if the portion of another tangent intercepted by TA , TB is bisected at its point of contact, prove that it is parallel to AB .

7. P, Q is any pair of conjugate points of a circle lying on a given line l ; O is the projection of the centre of the circle on the line l ; show that $PO.OQ$ is constant and equal to the square of the tangent to the circle from O .

8. Lines through a centre of similitude of two circles which are conjugate w. r. t. one circle are also conjugate to w. r. t. the other.

[Centres of similitude of two circles are the points which divide the line joining their centres internally and externally in the ratio of their radii].

9. V, V' are conjugate points w. r. t. a circle and PP' is a variable chord through V' ; prove that VP, VP' generate an involution pencil ; determine also the double lines of the pencil.

10. Show that the orthocentre of the harmonic triangle of a quadrilateral circumscribed to a circle is at the centre of the circle.

11. H is the orthocentre of a triangle ABC ; show that the polar circles of the triangles ABC, BCH, CAH, ABH are orthogonal in pairs.

Q. 7.8. **The Conic. Def.** The projection of a circle on a plane is defined to be a conic

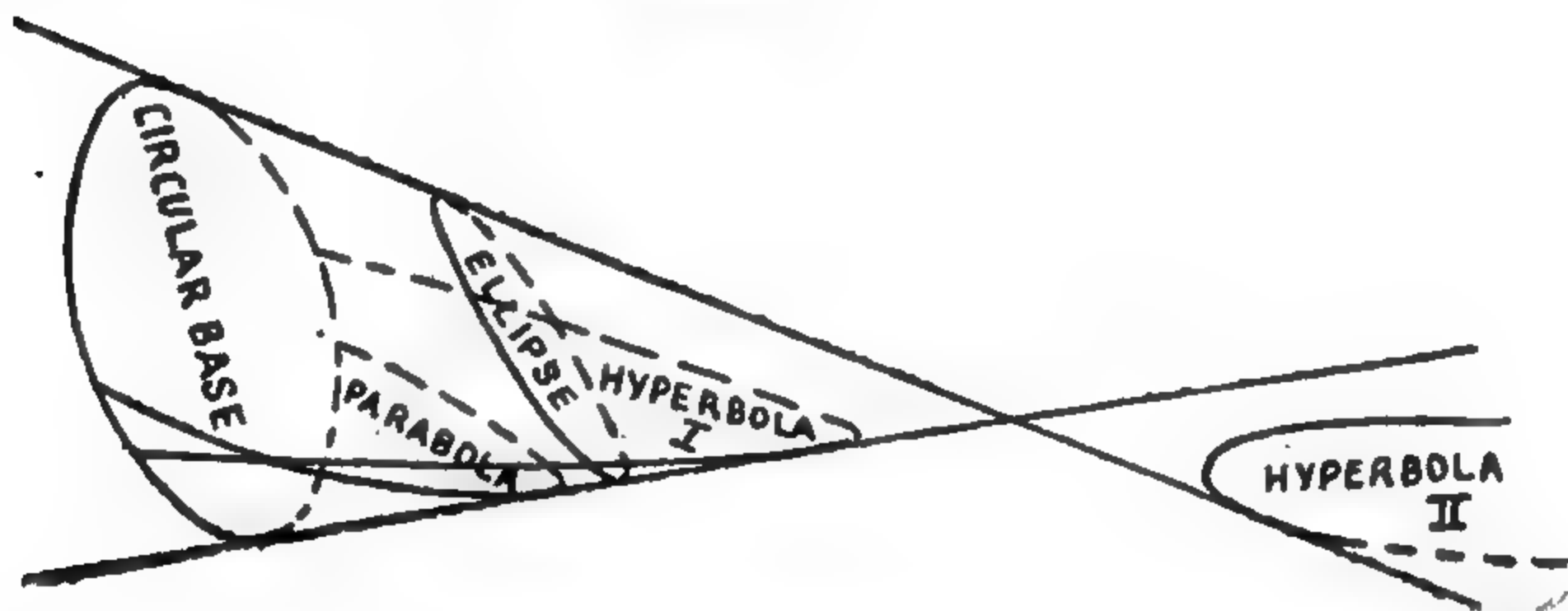


Fig. 64.

Consider a circle and *any* point V lying outside the plane α of the circle. Let V be joined to each point of the circle. The bundle of lines which we obtain in this manner determine a surface, called a circular cone. The lines are called the *generators* of the cone. Let these lines be extended indefinitely on either side of V . The projection of the circle on any plane π from the vertex V is essentially a section of this cone by the plane π . Consequently a conic is sometimes also called a *conic section*.

The conic sections are classified according as the vanishing line of the plane α for the vertex V intersects the circle in two real and distinct points, touches the circle, or cuts it in two imaginary points.

In the former case, the curve consists of two distinct branches corresponding to the two portions in which the vanishing line divides the circle, and is called a *hyperbola*; in the second case, it consists of a single branch extending to infinity and is called a *parabola*; and in the last case, it consists of a closed oval, and is called an *ellipse*.

Since the vanishing line meets the circle in two real and distinct points for the curve of projection to be a hyperbola, we say a hyperbola meets the line at infinity in two real points, these being the projections of the points common to the circle and the vanishing line. A similar reasoning will show that a parabola has two coincident points common with the line at infinity i. e., the line at infinity is a tangent to the parabola and an ellipse has two imaginary points common with the line at infinity.

Thus a conic is a hyperbola, a parabola or an ellipse according as it meets the line at infinity in its plane in a pair of real, distinct; coincident, or imaginary points.

Considered as sections of a circular cone, the conic is an ellipse, if the plane of section meets all the generators on the same side of the vertex, no generator being parallel to the plane; it is a hyperbola if it meets

*The point need not lie on the normal to the plane through the centre of the circle i. e., the cone which we ultimately get may not be a right circular one.

some generators on one side and some on the other side of the vertex, no generator being parallel to the plane in this case either; finally, it is a parabola if the plane of section is parallel to one of the generators, the plane consequently meeting all the generators on the same side of the vertex, and the curve of section extending to infinity.

Two special cases arise: (1) When the plane of section π is parallel to α , the section will be a circle.

(ii) When the plane of section passes through the vertex of the cone, the section is a pair of intersecting straight lines. This is evidently a limiting case of a hyperbola and is said to be a *degenerate conic*.

781. The following properties are obvious for the conic:—

1. Every straight line in the plane of the conic meets the conic in two points, real, coincident or imaginary.

2. From any point two tangents can be drawn to a conic, the tangents may be real, coincident or imaginary.

Hence a conic is a curve of the 2nd degree and 2nd class.

782. The polar of a point w . r. t. a conic is defined, (as in the case of the circle), to be the locus of the fourth harmonics of the point for the two points in which any straight line through the given point meets the conic.

Theorem. A pole and polar for a circle project into a pole and polar for the conic which is the projection of the circle.

Let P be any point, and p , its polar w. r. t. a given circle.

Let any line l through P meet the circle in Q and S and p in R . We have

$$(PR, QS) = -1.$$

Denoting the projections by corresponding dashed symbols, we see that the line l' through P' will meet the conic in Q' and S' and p' in R' .

Since harmonic ranges project into harmonic ranges,

$$(P'R', Q'S') = -1$$

so that R' is the fourth harmonic of P' for Q' and S' .

Therefore p' , which is the locus of R' , is the polar of P' .

We have incidentally proved that "*The polar of a point w. r. t. a conic is a straight line.*"

7·83. Since a tangent to a circle projects into a tangent to the conic which is the projection of the circle, we have the following results :—

(i) If P lies outside the conic, its polar w. r. t. the conic coincides with the chord of contact.

(ii) If P lies on the conic, its polar is the tangent to the conic at P .

7·84. The theorem of § 7·1 is fundamental for our purposes. A number of theorems which are true for the circle are now seen to be true for the conic also. For example, we have :

1. *If the polar of P passes through Q , the polar of Q passes through P . Two such points are said to be conjugate points for the conic.*

2. *Polars of collinear points are concurrent, the point of concurrence being the pole of the line on which the points lie.*

3. *Pairs of points on a straight line conjugate for a conic are in involution ; the united points of the involution being the points in which the line cuts the conic.*

4. *The meet of the polars of two points is the pole of their join.*

5. *If p contains the pole of q , q contains the pole of p . Two such lines are said to be conjugate lines for the conic.*

6. *Poles of concurrent lines are collinear; the line of collinearity being the polar of the point through which the lines pass.*

7. *The join of the poles of two lines is the polar of their meet.*

8. *Pairs of lines through a point conjugate w.r.t. a conic are in involution; the united rays of the pencil being the tangents from the vertex to the conic and conversely.*

9. *Self-conjugate triangles exist for a conic and of the three vertices of such a triangle, two must lie outside and one inside the conic.*

7·85. The centre of a conic is defined as the pole of the line at infinity in the plane of the conic. Thus the centre of a conic is the projection of the pole of the vanishing line for the circle whose projection is the conic.

Since the line at infinity touches a parabola, the centre of a parabola lies at infinity.

The pole of the vanishing line when it is not a tangent to the circle, is a point outside it, and will therefore project into a point lying in the finite part of the plane. Thus the ellipse and the hyperbola have a centre in the finite part of the plane: These two curves are, consequently, called *central conics* to distinguish them from the parabola.

Since the fourth harmonic of the centre for any two points in which a line through it meets the conic lies at infinity, every line through the centre is bisected at it.

Def. A line through the centre of a conic is called a *diameter*.

Since the centre of a parabola is at infinity, all diameters of a parabola are parallel.

Again, the pole of a diameter of a conic is a point at infinity. Hence tangents at the ends of a diameter are parallel.

7·86. **Conjugate Diameters.** Conjugate lines through the centre are said to be *conjugate diameters*.

Theorem. *The locus of the middle points of a system of parallel chords of a conic is a diameter conjugate to the diameter parallel to the chords.*

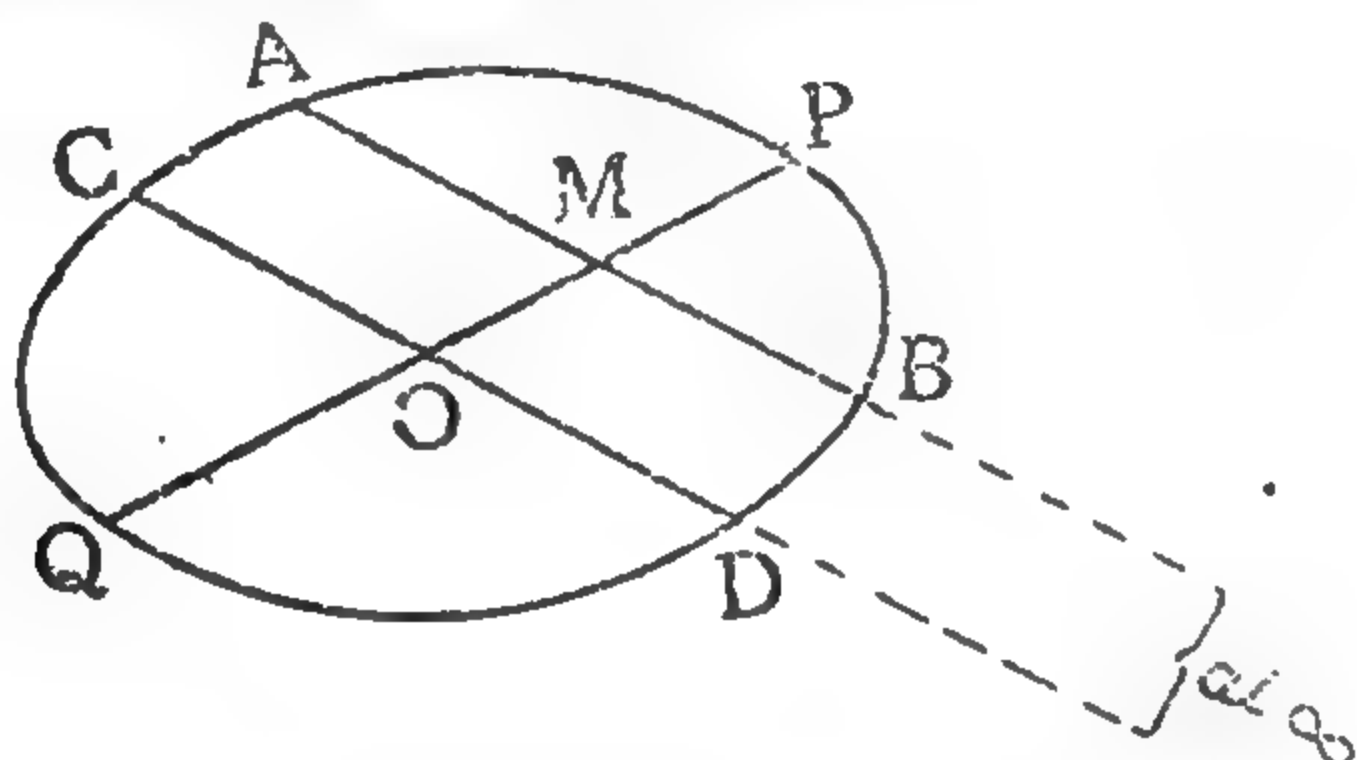


Fig. 65.

Let PQ , CD be a pair of conjugate diameters.

We have to show that every chord parallel to CD is bisected by PQ and *vice versa*.

The diameter PQ is the polar of W , the point at infinity on CD .

Let AB be any chord parallel to CD so that it passes through W . Let it meet the polar PQ of W in M .

$$\therefore (WM, AB) = -1$$

i.e. M is the middle point of AB .

Thus any chord parallel to CD is bisected by PQ .

Similarly, any chord parallel to PQ is bisected by CD .

Hence the theorem.

Cor. 1. Tangents at the ends of a system of parallel chords parallel to a diameter intersect on the conjugate diameter.

Let AB be any chord parallel to the diameter CD and let the diameter PQ be conjugate to CD .

Since the pole W of PQ lies on AB , the pole of AB which is the point of intersection of the tangents at A and B lies on PQ .

Cor. 2. Tangents at the ends of a diameter are parallel to the conjugate diameter.

This follows from the fact that the point of intersection T of the parallel tangents at the extremities P, Q of a diameter PQ is the pole of PQ so that T lies on the diameter conjugate to PQ .

Cor. 3. The join of the centre to the point of intersection of tangents at the ends of a chord bisects the chord.

787. Axes. Pairs of conjugate diameters of a conic determine an involution at the centre. Now, to every involutive pencil, there belongs one pair of orthogonal corresponding rays unless the involution pencil is orthogonal. The orthogonal pair of conjugate diameters of a conic are called its **principle axes** or simply the **axes** of the conic. We now prove the

Theorem *If the involution pencil of conjugate diameters of a conic is orthogonal the conic must be a circle.*

Consider any pair of conjugate diameters PQ, CD .

Since the tangents at the ends of a diameter are parallel to the conjugate diameter, therefore the tangents to the conic at P and Q are parallel to CD , and hence perpendicular to PQ .

Hence the tangent to the conic at any point being perpendicular to the line joining the point to the centre, the conic must be a circle.

Cor. 1. *Every conic through the circular points must be a circle.*

This follows from the fact that the pairs of conjugate points on the line at infinity determine an orthogonal involution pencil at the centre of the conic.

Cor. 2. *A conic cannot have more than one pair of orthogonal conjugate diameters.*

788. Asymptotes. **Def.** Tangents from the centre of a conic are defined to be its *asymptotes*.

From the definition, it follows that

the asymptotes are the united rays of the involution pencil of conjugate diameters.

Again, the principle axes, being a pair of conjugate diameters, are harmonic *w r. t.* the asymptotes. Moreover, they (axes) are at right angles. Hence they, must bisect the angles between the asymptotes.

Since the centre of a conic is the projection of the pole P of the vanishing line, the asymptotes are the projections of the tangents from P to the circle and are real only when these tangents are real *i.e.*, when P lies outside the circle, so that its polar, the vanishing line, meets the circle, in two real and distinct points. We know also that the projection of the circle is, in that case, a hyperbola.

Hence the asymptotes are real and finite only in the case of a hyperbola.

The points of contact of tangents from P having been projected to infinity, the asymptotes touch the conic at infinity. Hence an asymptote is sometimes defined as *the tangent whose point of contact is a point at infinity, but which does not lie wholly at infinity.*

7.9. We close this chapter with two important theorems. These theorems could have been proved for a circle and then with the help of projection, they could be seen to be true for a conic. We, however, prove them independently of any consideration of circle and leave the method of projection to the student to develop.

7.91 Theorem. *The harmonic triangle of a quadrangle inscribed in a conic is self-conjugate for the conic.*

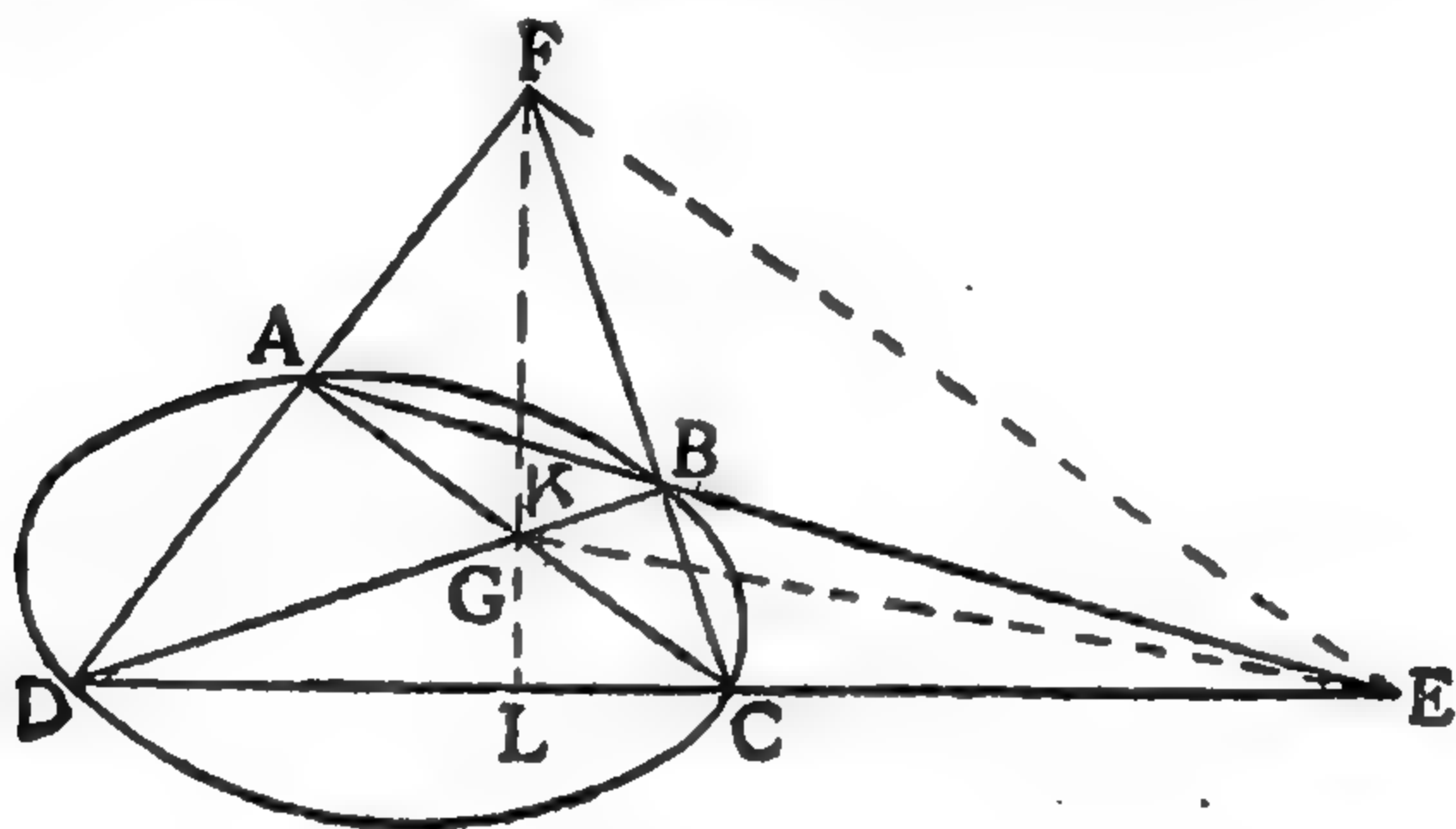


Fig. 66

Let $ABCD$ be a quadrangle inscribed in a conic. We have to show that the harmonic triangle EFG of the quadrangle is self-conjugate for the conic.

Let FG meet AB and CD in K and L respectively

Then, from the harmonic property of the quadrangle, we have

$$F(AB, EG) = -1.$$

$$\therefore (AB, EK) = -1 \text{ and } (DC, EL) = -1.$$

so that K is the fourth harmonic of E for A and B and L is the fourth harmonic of E for C and D .

$$\therefore KL \text{ i. e., } FG \text{ is the polar of } E.$$

Similarly, it can be shown that GE is the polar of F and EF is the polar of G .

Hence EFG is a self-conjugate triangle for the conic.

7.91. The above theorem suggests an easy method for the construction of the polar of a given point for a conic and of the construction of tangents from a given point.

Through the given point E , draw any two lines meeting the conic in four points A, B, C, D . Then the side of the harmonic triangle of this quadrangle opposite to E is the polar of E .

To obtain the tangents from E , join E to the points where its polar meets the conic.

7.92 Theorem. *The harmonic triangle of a quadrilateral circumscribed to a conic is self-conjugate for the conic.*

Let the four tangents be a, b, c, d . We have to prove that the harmonic triangle PQR of the quadrilateral is self-conjugate for the conic.

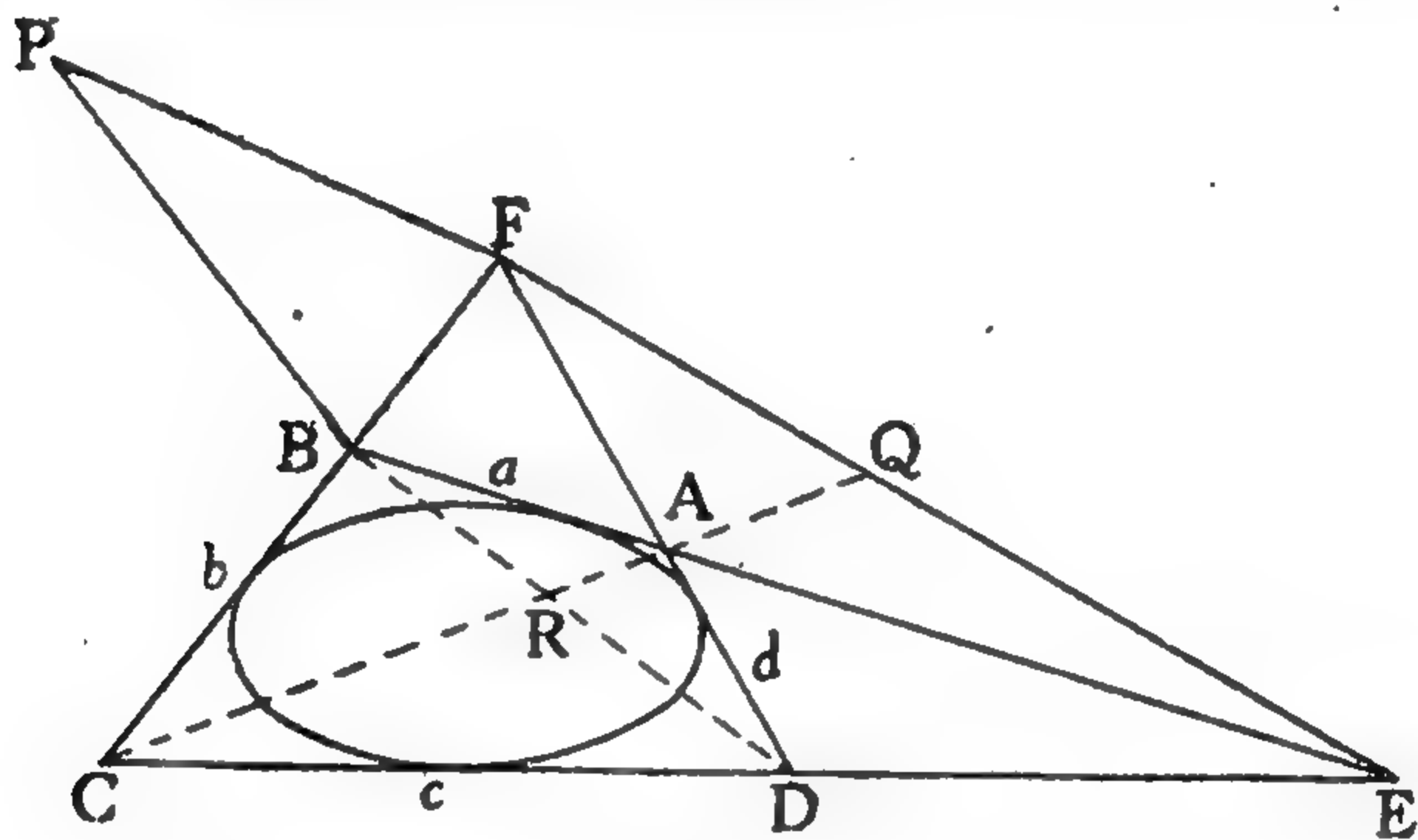


Fig.

Let A, B, C, D, E, F be the vertices of the quadrilateral.

Since ERF is the harmonic triangle of the quadrangle $ABCD$, we have

$$F(RE, CD) = -1.$$

But FC, FD are tangents to the conic.

Therefore FR, FE is a pair of conjugate lines for the conic through F . i. e., FR contains the pole of FE .

Similarly, considering the pencil at E , we conclude that ER contains the pole of EF .

Hence R is the pole of EF i. e., of PQ .

It can similarly be shown that P is the pole of QR and Q is the pole of RP .

Hence PQR is a self-conjugate triangle for the conic.

Ex. Show that EFR is a self conjugate triangle for the conic in the above diagram and deduce that the harmonic triangle of a quadrangle inscribed in a conic coincides with that of the quadrilateral formed by tangents at the four given points.

[$\therefore FR$ is the polar of E , FR must pass through the points of contact of tangents from E . Similarly, ER passes through the points of contact of tangents from F].

Ex. 1. *A conic projects into a conic.*

Let C be a given conic in a plane α . Let C' be the projection of C in the plane π .

(i) Draw any line l' in the plane π .

The points of intersection of l' and C' are the projections of the points in which l (whose projection is l' in the plane π) meets C . But l meets C in two points.

Therefore l' meets C' in two points.

(ii) Let P' be any point in π plane. The tangents from P' to C' are the projections of the tangents which can be drawn to C from P (whose projection is P').

But only two tangents can be drawn from P to C .

Hence only two tangents can be drawn from P' to C' .

We find then that the projection of a conic is a curve of the second degree and second class and hence must be a conic.

Ex. 2. *A conic can be projected into a circle.*

Let C be a given conic in the plane α .

Let l be the vanishing line of the plane α and L the pole of l w. r. t. C .

Let $p, p_1; q, q_1$, be two pairs of conjugate lines for C through L . Project l to infinity and, at the same time, the angles between $p, p_1; q, q_1$ into right angles.

Denoting projections by corresponding dashed symbols, we observe that l' is the line at infinity in π .

$\therefore L'$ is the centre of C' , the projection of C

p, p_1, q_1, q project into a pair of perpendicular diameters.

And since $p, p_1; q, q_1$ are conjugate pairs of lines through L for C , therefore

$P', p'_1; q', q'_1$ are conjugate diameters of C' .

Therefore two pairs of conjugate diameters of C' are perpendicular, so that every pair of conjugate diameters must be perpendicular.

Hence C' is a circle.

Ex. 3. Show that concurrent chords of a circle, can be projected into a system of parallel chords of the conic which is the projection of the circle.

Hence obtain the locus of the middle points of a system of parallel chords of the conic.

Let Q be the point of concurrence of the chords $AA', BB', \dots$ of a given circle.

Let p be any line through Q . Let q be the polar of Q and P the pole of p .

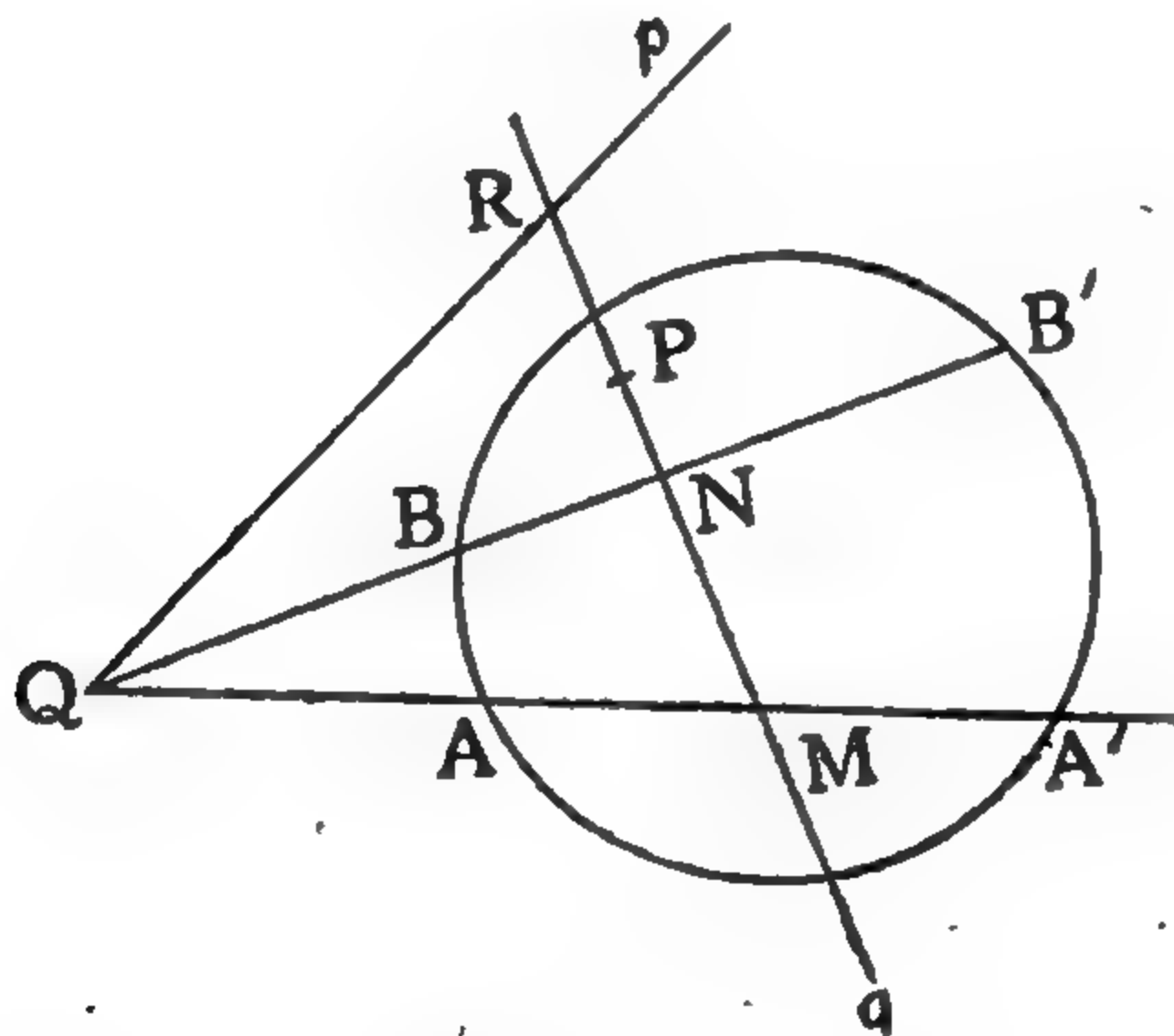


Fig. 68

Project the circle so that p projects to infinity.

Denote the projections by corresponding dashed symbols.

Since p' is the line at infinity in the plane of projection, its pole P' , the projection of P , is the centre of the conic.

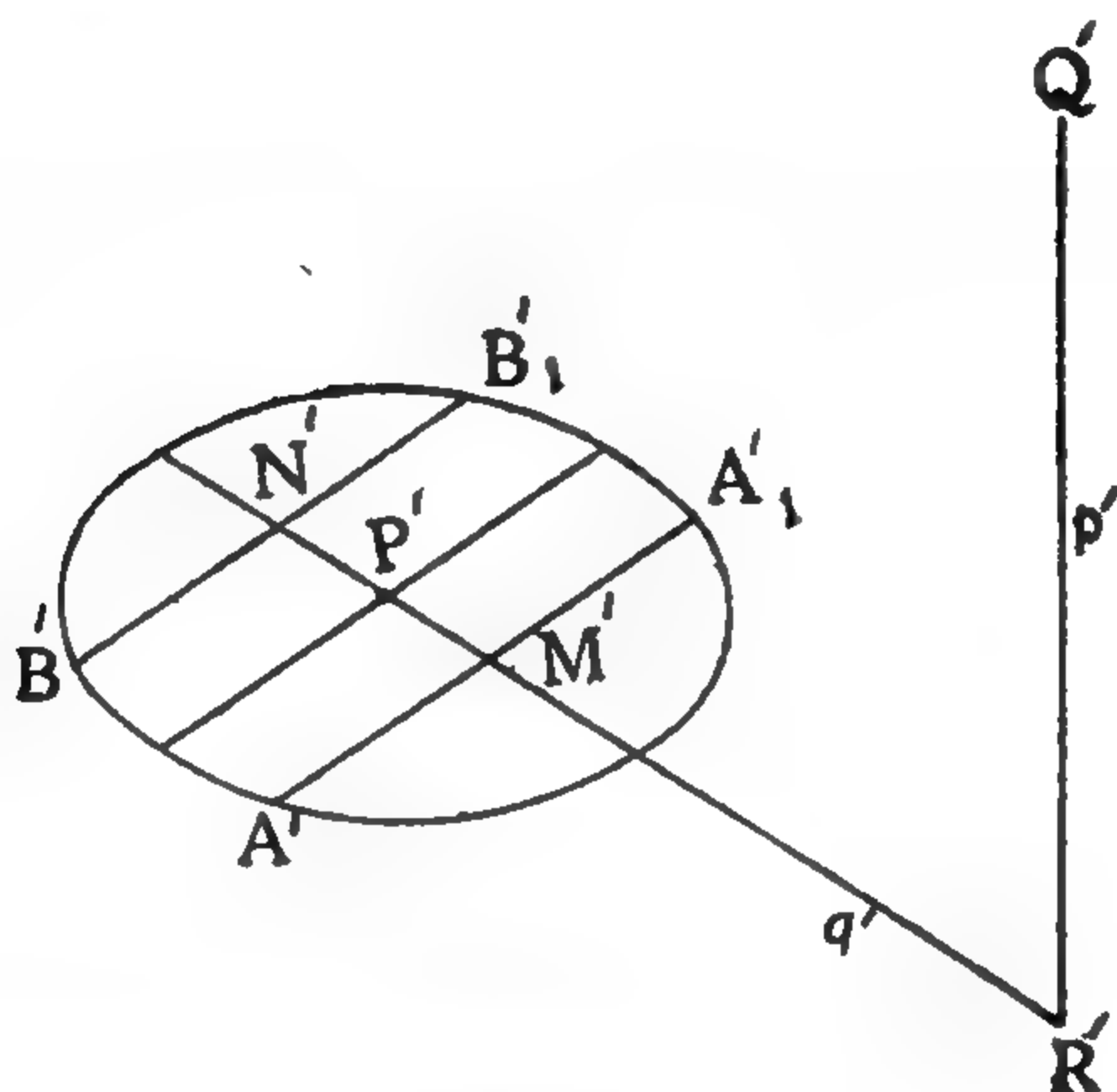


Fig. 69

Also, Q' is a point at infinity so that $A'A'_1, B'B'_1, \dots$ are parallel.

Thus all chords of the circle through the point Q project into parallel chords of the conic.

Again, $P'Q', P'R'$ are conjugate diameters of the conic, for, PQ, PR are conjugate lines for the circle.

M, N are the fourth harmonics of $A, A'; B, B'$ respectively for Q .

$\therefore M', N'$ are the fourth harmonics of $A', A'_1; B', B'_1$, respectively for Q' .

But Q' is at infinity. $\therefore M', N'$ are the mid-points of $A'A'_1, B'B'_1$, respectively.

$\therefore M', N'$ lie on q' , the diameter of the conic conjugate to $P'Q'$.

Hence the mid-points of all chords parallel to $P'Q'$ lie on the diameter conjugate to $P'Q'$.

7.93. Theorem. *If a quadrilateral is such that two pairs of opposite vertices are conjugate for a conic, then the third pair will also be conjugate for the conic.*

(See fig. 45, P. 94.)

Let $A, A' : B, B' : C, C'$ be the opposite pairs of vertices of the quadrilateral and let $A, A' : B, B'$ be the pairs of conjugate points for a conic.

Let T be the pole of the line AB .

Now the polar of A passes through T and also through A' so that TA' is the polar of A . Therefore TA, TA' are conjugate lines for the conic.

Similarly TB, TB' are conjugate lines for the conic.

By theorem of §, 682, P. 93 TC, TC' , belong to the involution determined by $TA, TA' : TB, TB'$ i. e. TC, TC' belong to the involution determined by the conjugate lines through T .

$\therefore TC, TC'$, are conjugate lines for the conic.

Thus, the pole of the line TC which is conjugate of AB as well as TC' is the point X where TC' meets AB .

Again, the pole of the point C which is conjugate of T as well as X is the line TX which passes through C' , so that C, C' are conjugate points for the conic.

CHAPTER VIII

CO-AXAL CIRCLES

8.01. In this chapter we shall study the properties of a co-axal system of circles. The discussion will be, more or less, of an elementary nature.

8.02. **Power of a point.** In chapter, VII we have defined the power of a point P w. r. t. a circle as the square of the length of the tangent PT from P to the circle. From elementary geometry, we know that if PAB is any secant of the circle through P meeting the circle in A and B , then $PA \cdot PB = PT^2$. Hence the power of a point w. r. t. a circle can also be defined as *the rectangle contained by the segments of any secant through the point*. A point can, consequently, be said to lie outside, on or inside a circle according as its power w. r. t. the circle is positive, zero or negative.

The power of a point w. r. t. two circles is the same if the tangents from the point to the circles are equal in length.

Ex. Show that the power of a point on the common chord of two circles is the same for both the circles.

8.2. **Theorem.** *The locus of a point whose powers w. r. t. two circles are equal is a straight line perpendicular to their line of centres.*

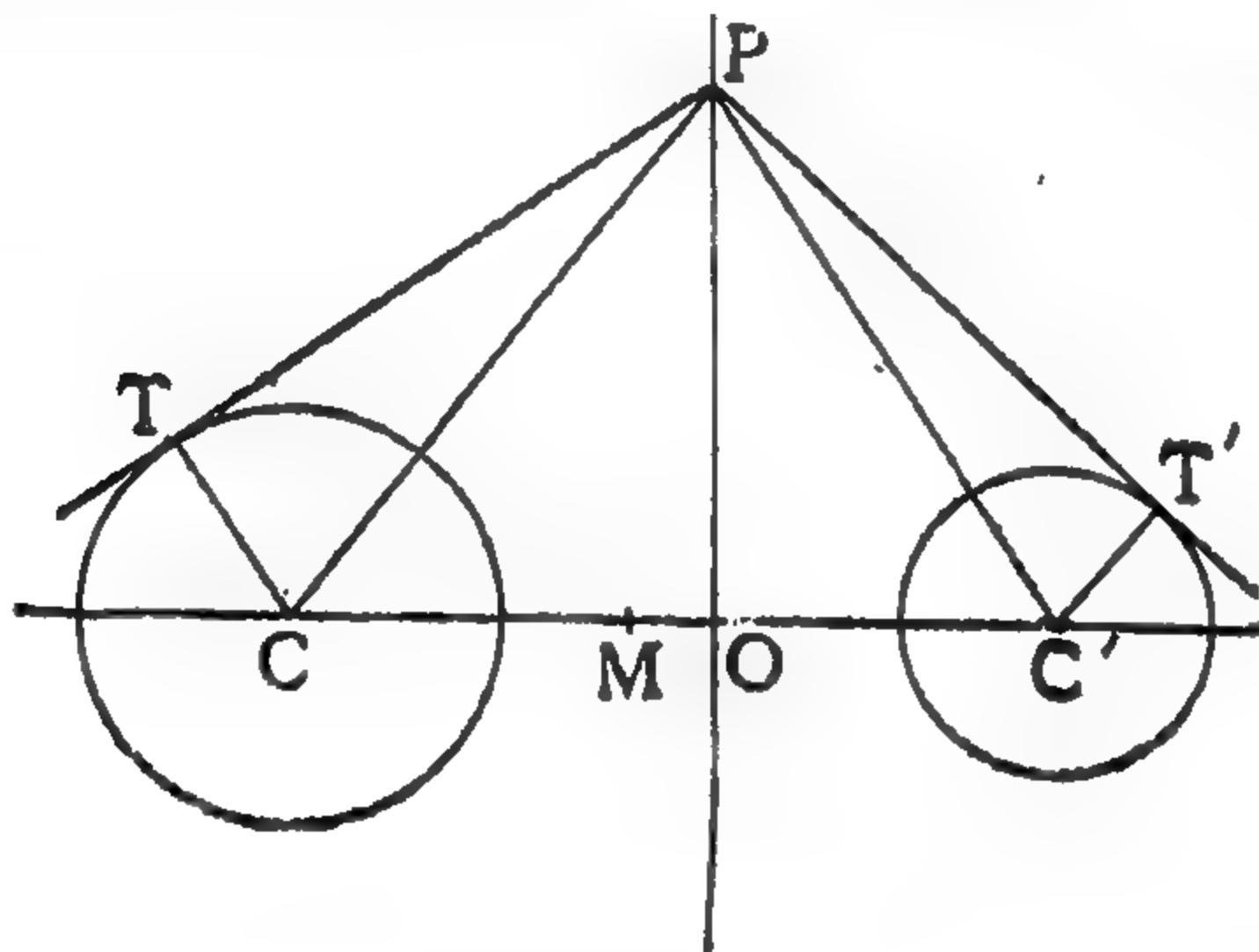


Fig. 70

Let P be any point on the locus so that $PT = PT'$.

Let PO be perpendicular from P to the line of centres CC' . We will show that O is a fixed point.

$$\text{Now } PT^2 = PC^2 - CT^2 = CO^2 + OP^2 - CT^2.$$

$$\text{and } PT'^2 = PC'^2 - C'T'^2 = C'O^2 + OP^2 - C'T'^2.$$

$$\text{Also } PT = PT',$$

$$\therefore CO^2 + OP^2 - CT^2 = C'O^2 + OP^2 - C'T'^2.$$

$$\text{or } CO^2 - C'O^2 = CT^2 - C'T'^2$$

$$\text{or } (CO - C'O)(CO + C'O) = CT^2 - C'T'^2$$

$$\text{or } (CO + OC')(CO - OC') = CT^2 - C'T'^2$$

$$\text{or } CC' \cdot (CM + MO - MC' + MO) = CT^2 - C'T'^2$$

$$\text{or } 2CC' \cdot MO = CT^2 - C'T'^2, \text{ where } M \text{ is the middle point of } CC'.$$

Now CC' , CT and $C'T'$ are constant quantities and M is a fixed point. Therefore MO is constant

Hence O is a fixed point.

The locus of P is, therefore, a straight line perpendicular to CC' .

Def. The locus of a point whose power w. r. t. two circles is the same is called the **Radical-Axis** of the two circles.

Cor. 1. The radical axis of two intersecting circles is their common chord and of two touching circles it is the common tangent at the point of contact.

Cor. 2. The radical axis of two circles bisects their common tangents.

8.22. Theorem. *The three radical axes of three circles taken in pairs are concurrent.*

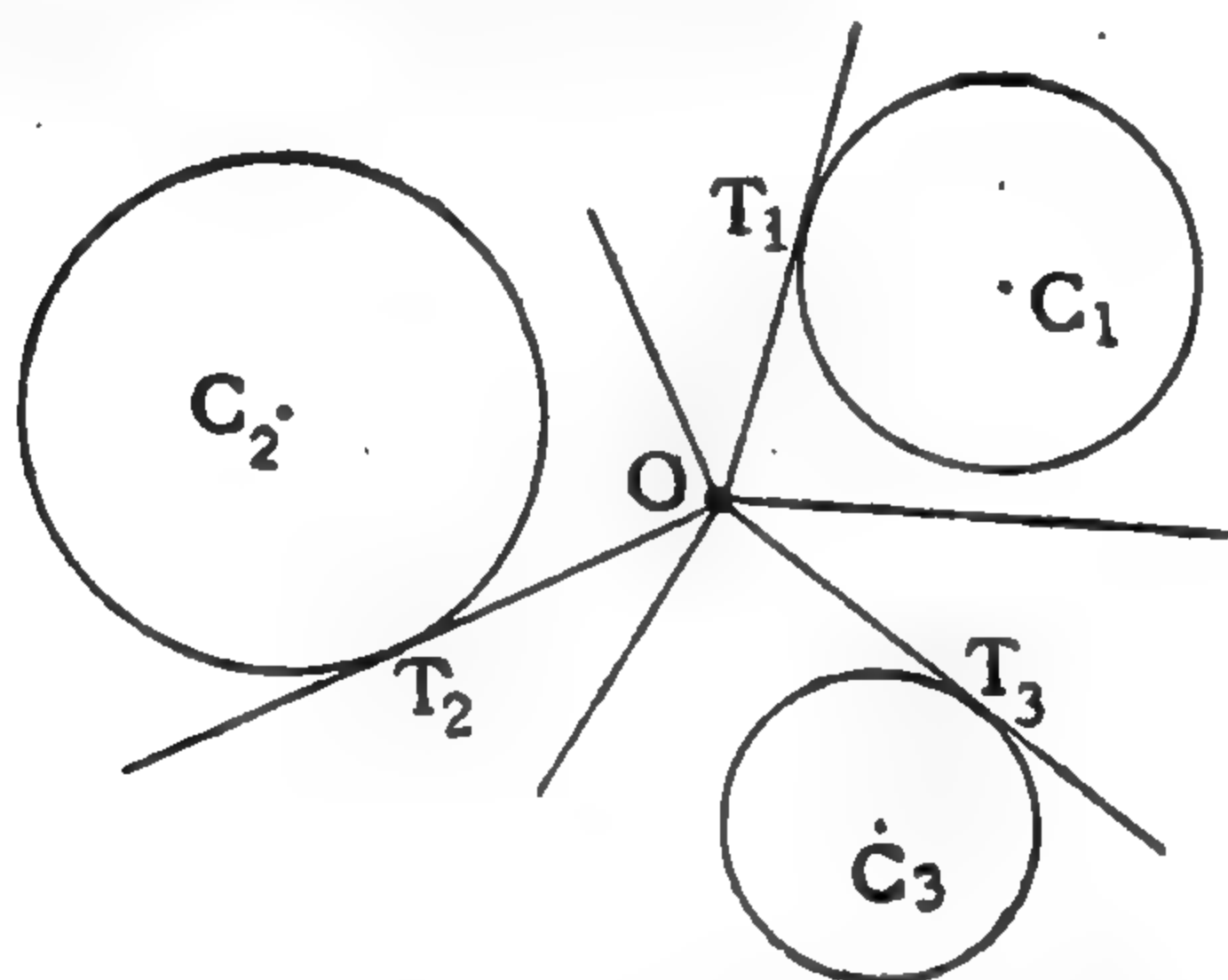


Fig. 71

Let the radical axis of S_1 and S_2 meet the radical axis of S_2 and S_3 in O . We have to show that O lies on the radical axis of S_3 and S_1 also.

Since O lies on the radical axis of S_1 and S_2 , therefore
Power of O w. r. t. $S_1 =$ Its power w. r. t. S_2 , ...(1)

Again, since O lies on the radical axis of S_2 and S_3 ,
therefore

Power of O w. r. t. $S_2 =$ Its power w. r. t. S_3 , ...(2)

From (1) and (2), we have

Power of O w. r. t. $S_1 =$ Its power w. r. t. S_3 .

Therefore O is a point on the radical axis of S_3 and S_1 ,

The three radical axes are, therefore, concurrent.

Def. The point of concurrence of the radical axes of three circles taken in pairs is called their *Radical centre*.

The lengths of the tangents from the radical centre to each of these circles being equal, if a circle is drawn with its centre at this point and radius equal to the length of the tangent from this point to any of the circles, it must cut all the three circles orthogonally.

Conversely, a circle, which cuts each of three given circles orthogonally, must have its centre on the radical axes of the three circles taken in pairs i.e., at their radical centre. Moreover, the radius of such a circle is also fixed. Hence there is one and only one circle which cuts three given circles orthogonally.

8.23. *To construct the radical axis of two circles.*

Draw a circle meeting the two given circles S_1, S_2 in real points. Let the common chords of S and S_1 meet the common chord of S and S_2 in P . Then P , being the radical centre of the three circles, is on the radical axis of S_1 and S_2 . Similarly, by drawing another circle S' to meet S_1 and S_2 in real points, we could obtain another point P' on the radical axis.

The join of P and P' is the radical axis required.

8.24. **Theorem.** *The difference of the powers of a point w. r. t. two given circles is proportional to the length of the perpendicular from the point to their radical axis.*

Let A be a point in the plane of the given circles S_1 and S_2 .

Let AT_1, AT_2 be the tangents from A to the circles S_1 and S_2 and let AD be the perpendicular from A to their radical axis.

We will show that $AT_1^2 - AT_2^2$ varies as AD .

Drop AF perpendicular to C_1C_2 .

Let M be the mid-point of C_1C_2 .

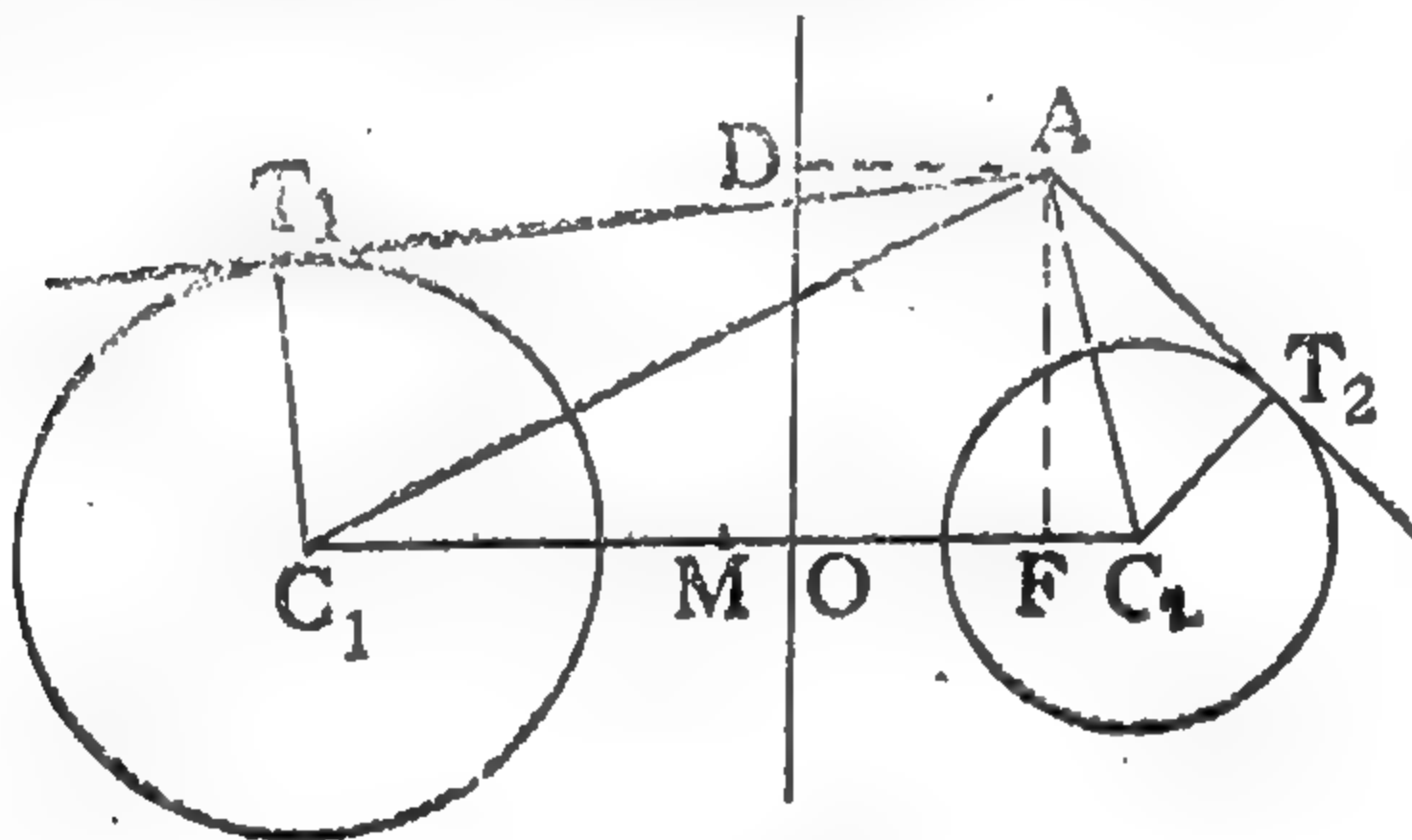


Fig. 72

$$AT_1^2 - AT_2^2 = (AC_1^2 - C_1T_1^2) - (AC_2^2 - C_2T_2^2)$$

$$\begin{aligned}
&= (C_1 F^2 + F A^2 - C_1 T_1^2) - (C_2 F^2 + F A^2 - C_2 T_2^2) \\
&= (C_1 F^2 - C_2 F^2) - (C_1 T_1^2 - C_2 T_2^2) \\
&= (C_1 F - C_2 F) (C_1 F + C_2 F) - (C_1 T_1^2 - C_2 T_2^2) \\
&= (C_1 F + F C_2) (C_1 F - F C_2) - 2C_1 C_2 \cdot MO. [\S 8.2]. \\
&= 2C_1 C_2 \cdot MF - 2C_1 C_2 \cdot MO \\
&= 2C_1 C_2 (MF - MO) \\
&= 2C_1 C_2 \cdot OF \\
&= 2C_1 C_2 \cdot DA
\end{aligned}$$

Hence the theorem.

8'31. *Every circle which cuts two circles orthogonally must have its centre on the radical axis; for the power of the centre of such a circle w.r.t. each of the two circles must be the same, being the square of the radius of this circle itself.*

The converse, however, is not true. A circle with its centre on the radical axis of two circles may not cut the circles orthogonally, because the radius of this circle may not be equal to the length of the tangent from the centre to each of the two given circles. It should, however, be noted that if a circle has its centre on the radical axis and cuts one of the two circles orthogonally, it must cut the other also orthogonally.

8'32. **Theorem.** *Every circle which cuts two circles orthogonally meets their line of centres in a fixed pair of points.*

Let the line of centres $C_1 C_2$ of the two given circles meet them in $A, B; P, Q$ and the third orthogonal circle in L_1, L_2 .

Since an orthogonal circle must cut every diameter of the other circle in a pair of inverse points, we have,

$(L_1 L_2, AB) = -1$ and $(L_1 L_2, PQ) = -1$,
so that L_1, L_2 are harmonically conjugate w. r. t. each of the two fixed pairs of points $A, B; P, Q$ and therefore L_1, L_2 is a fixed pair of points.

Conversely, every circle through L_1, L_2 must cut each of the two circles orthogonally, for the two points are a pair of inverse points for each of the two circles.

Note. L_1, L_2 are the united points of the involution determined by the two pairs of points $A, B; P, Q$; also, the point O where the line of centres meets their radical axis is the centre of this involution; L_1, L_2 are real only if both the points of either pair lie on the same side of O i.e., if the two circles do not intersect in real points. Thus L_1, L_2 are real or imaginary according as the two circles intersect in imaginary or real points. It should also be noted that the radical axis bisects $L_1 L_2$.

8.33. Theorem. *The polars with respect to two circles of any point on their radical axis intersect on the radical axis.*

Let P be any point on the radical axis of two circles S_1, S_2 . Let $PT_1, PT_1'; PT_2, PT_2'$ be tangents from P to the two circles. Let $T_1 T_1'$ meet $T_2 T_2'$ in P' . We have to show that P' is a point on the radical axis.

Draw a circle with centre P and radius equal to PT_1 .

Since P is a point on the radical axis, tangents from P to the two circles are equal, and therefore the circle with centre P and radius PT_1 will pass through T_1', T_2, T_2' also, so that $T_1 T_1', T_2 T_2'$ which meet in P' are chords of the same circle.

$\therefore P'T_1'. P'T_1 = P'T_2'. P'T_2$, i.e., the powers of P' w. r. t. the two circles are equal so that P' is a point on the radical axis.

Aliter. We know that a circle which cuts each of the two given circles orthogonally, has its centre on the radical axis. Let such a circle meet the radical axis in P, P' . Then P, P' are conjugate points w. r. t. both the circles. Hence the proposition.

8.4. Coaxal circles. *A system of circles is said to be co-axial if any two circles of the system have the same radical axis.*

Thus, if two circles of the system intersect in real points, every circle of the system will pass through their points of intersection and the co-axal system, in this case, is said to be of an *intersecting type*. (Fig. 73).

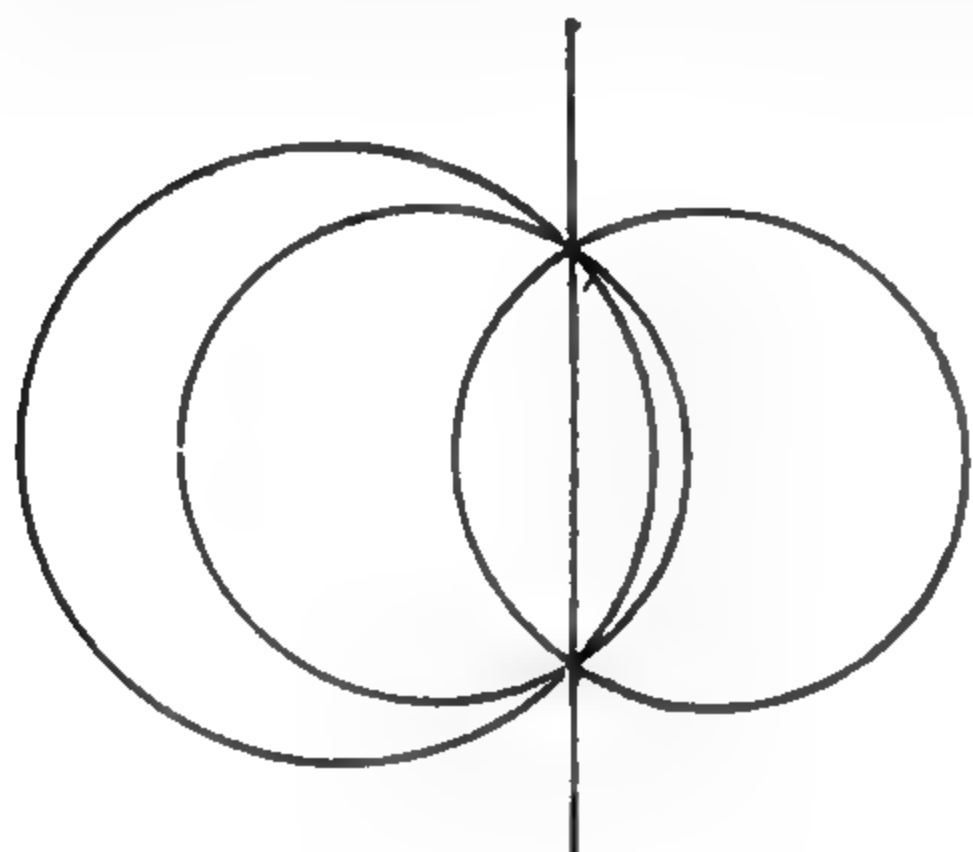


Fig. 73

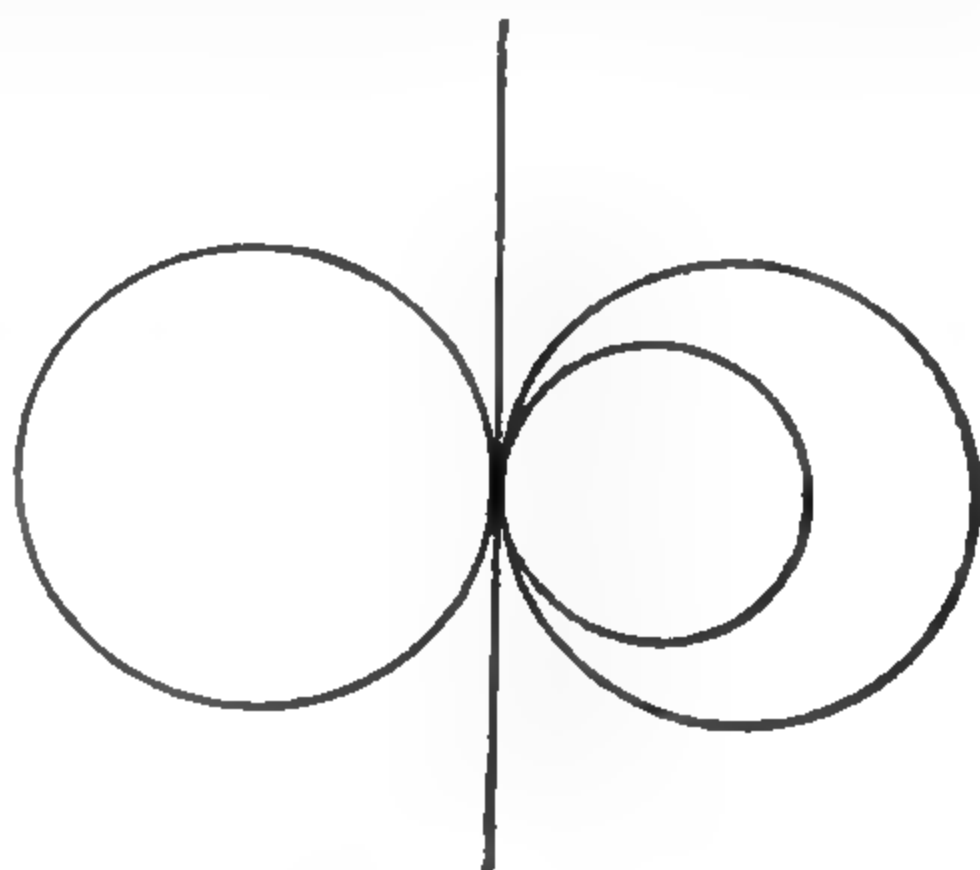


Fig. 74

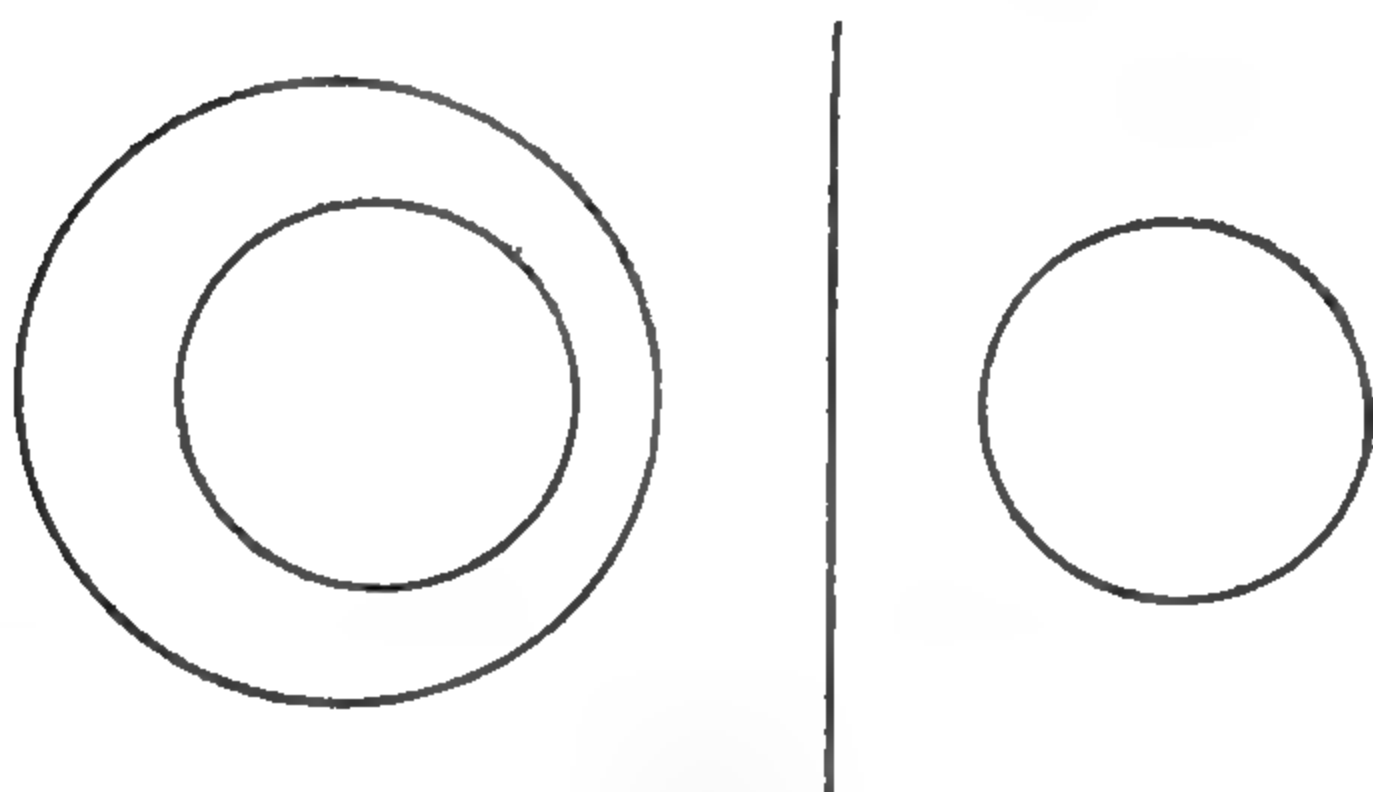


Fig. 75

As a limiting case of the above, if the two points of intersection coincide, i.e., two circles of the system touch, every two circles of the system will touch at this point. (Fig. 74).

If no two circles of a co-axal system intersect, then it is said to be of a *non-intersecting type*. (Fig. 75).

8.41. Properties of co-axal circles.

1. Since the line of centres of two circles is perpendicular to their radical axis, the centres of the circles of a co-axal system are collinear.

2. Only one circle of a co-axal system can be drawn so as to pass through a given point.

Every circle of the system passes through the two points, real, coincident or imaginary, common to all circles of the system so that a third point fixes the circle uniquely.

Circles of a co-axal system are, consequently, sometimes spoken of as a *Pencil of circles*.

3. Two circles of a co-axal system can be drawn so as to touch a given line.

Let the given line meet the radical axis in O and circles of the system in pairs of points,

$$A_1, A_2; B_1, B_2; C_1, C_2, \dots\dots\dots$$

The power of O , w. r. t. each of the circles being same, we have

$$OA_1 \cdot OA_2 = OB_1 \cdot OB_2 = OC_1 \cdot OC_2 = \dots\dots\dots$$

Hence $A_1, A_2; B_1, B_2; C_1, C_2, \dots\dots\dots$ are pairs of points in an involution whose centre is at O .

Let U', U'' be the united points of this involution.

$$\therefore OA_1 \cdot OA_2 = OB_1 \cdot OB_2 = \dots\dots\dots = OU'^2 = OU''^2.$$

Let a circle of the system through U' meet, if possible, the given line in another point, say, U . Then U, U' being a point-pair of the involution,

$$OU \cdot OU' = OU'^2 \text{ i.e., } OU = OU'$$

i.e., U coincides with U' so that the circle touches the given line. Similarly, the circle of the system through U'' will touch the given line.

Thus the circles of the system which pass through U', U'' are the two required circles.

We have incidentally proved that a line cuts a system of co-axal circles in pairs of points belonging to an involution and the double points of this involution are conjugate points for every circle of the system.

The following results immediately follow from the above :—

(i) U', U'' are conjugate points for every circle of the system. Hence a circle on U', U'' , as diameter cuts every circle of the co-axal system orthogonally.

(ii) The polars of a point *w. r. t.* the circles of a co-axal system are concurrent.

Let U' be the given point. Draw the tangent at U' to the circle of the system through U' and let it meet the radical axis in O . Take a point U'' on this line such that $OU'' = U'O$, so that O is the mid-point of $U' U''$.

Then U', U'' are conjugate points for every circle of the system. Hence polars of U' for the circles of the system will ~~concur~~ concur in U'' .

8.5. Limiting points. *If a circle cuts two circles of a co-axal system orthogonally it cuts every circle of the system orthogonally.* for the power of its centre, (this being on the radical axis) for every circle of the system equals the square of its radius.

From § 8.32, it follows that all such circles meet the line of centres in the same two points which are real if the circles are of a non-intersecting type. These points are called the **limiting points** of the co-axal system. Evidently their distance from any point on the radical axis is the same as the length of the tangent from the point to any circle of the system. These points can, therefore, be looked upon as degenerate members of the co-axal system with radius zero and, as such, are sometimes called **Point Circles** of the system.

8.51. Properties of Limiting Points.

1. *The limiting points are conjugate w. r. t. every circle of the co-axal system.*

A circle which is orthogonal to every circle of the system cuts the line of centres in L' and L'' , the limiting points of the co-axals. Hence if the line of centres cuts any circle of the system in the points A_1, A_2 ,

$$(L' L'', A_1 A_2) = -1.$$

Hence L', L'' are conjugate w.r.t. every circle of the system.

Cor. 1. L', L'' are the united points of the involution determined by the co-axal system on the line of centres.

This could also be seen from the fact that

$$OA_1.OA_2 = OB_1.OB_2 = \dots = OL'^2 = OL''^2.$$

Cor. 2. Every circle through L', L'' cuts all the circles of the co-axal system orthogonally.

Cor. 3. Conjugate Systems of Co-axal Circles.

We have already seen that a circle which is orthogonal to all the circles of a co-axal system passes through the limiting points. Thus the system of circles which are orthogonal to every circle of a co-axal system, have a pair of points in common, viz., the limiting points of the original system, and as such, belong to a co-axal system, called the **conjugate system** of the given system. The two systems are such that every circle of one of the systems cuts every circle of the other orthogonally.

It will also be noted that the line of centres of one is the radical axis of the other and the common points of one are the limiting points of the other so that of a given system of co-axals and its conjugate, one must be of an intersecting type and the other of a non-intersecting type.

No circle of a system of co-axal circles can have its centre between the limiting points L', L'' .

Let any circle of the co-axal system having its centre at C be cut by the line of centres in A_1, A_2 . Then

$$(A_1 A_2, L' L'') = -1.$$

and since C is the middle point of $A_1 A_2$,

$$CL'.CL'' = CA_1^2.$$

$\therefore CL', CL''$ must be measured in the same sense i.e., C must lie outside $L' L''$.

3. If a line touches two members of a co-axal system in T_1, T_2 , then T_1T_2 subtends a right angle at either limiting point.

Let T_1T_2 meet the radical axis in P .

Then, $PT_1 = PT_2 =$ the length of the tangent from P to any circle of the system $= PL' = PL''$.

In the triangle $T_1L'T_2$, the median $L'P =$ half the opposite side.

$\therefore \angle T_1L'T_2 = 1 \text{ rt. angle.}$

Similarly, $\angle T_1L''T_2 = 1 \text{ rt. angle.}$

Cor. If Q_1, Q_2 be a pair of points conjugate w.r.t. every circle of a co-axal system, then Q_1Q_2 subtends a right angle at either limiting point.

Exercise

Ex. 1. Show that the radical axes of the circles of a co-axal system and any given circle are concurrent.

Let S be the given circle. Consider any circle S_1 of the co-axal system and keep it fixed. Let S_2 be any variable circle of the co-axal system. The three radical axes of S, S_1 and S_2 taken in pairs are concurrent, i. e., the radical axis of S and S_2 passes through the fixed point of intersection of the radical axis of S, S_1 and of the common radical axis of the co-axal system. Hence the proposition.

8.6. Theorem. The circles on the diagonals of a complete quadrilateral as diameters are co-axal.

Let AC, BD, EF be the diagonals of a complete quadrilateral. We have to show that the circles on these diagonals as diameters are co-axal.

Let PQR be the harmonic triangle of the quadrilateral.

Let the circle on AC and BD as diameters intersect in V, V' .

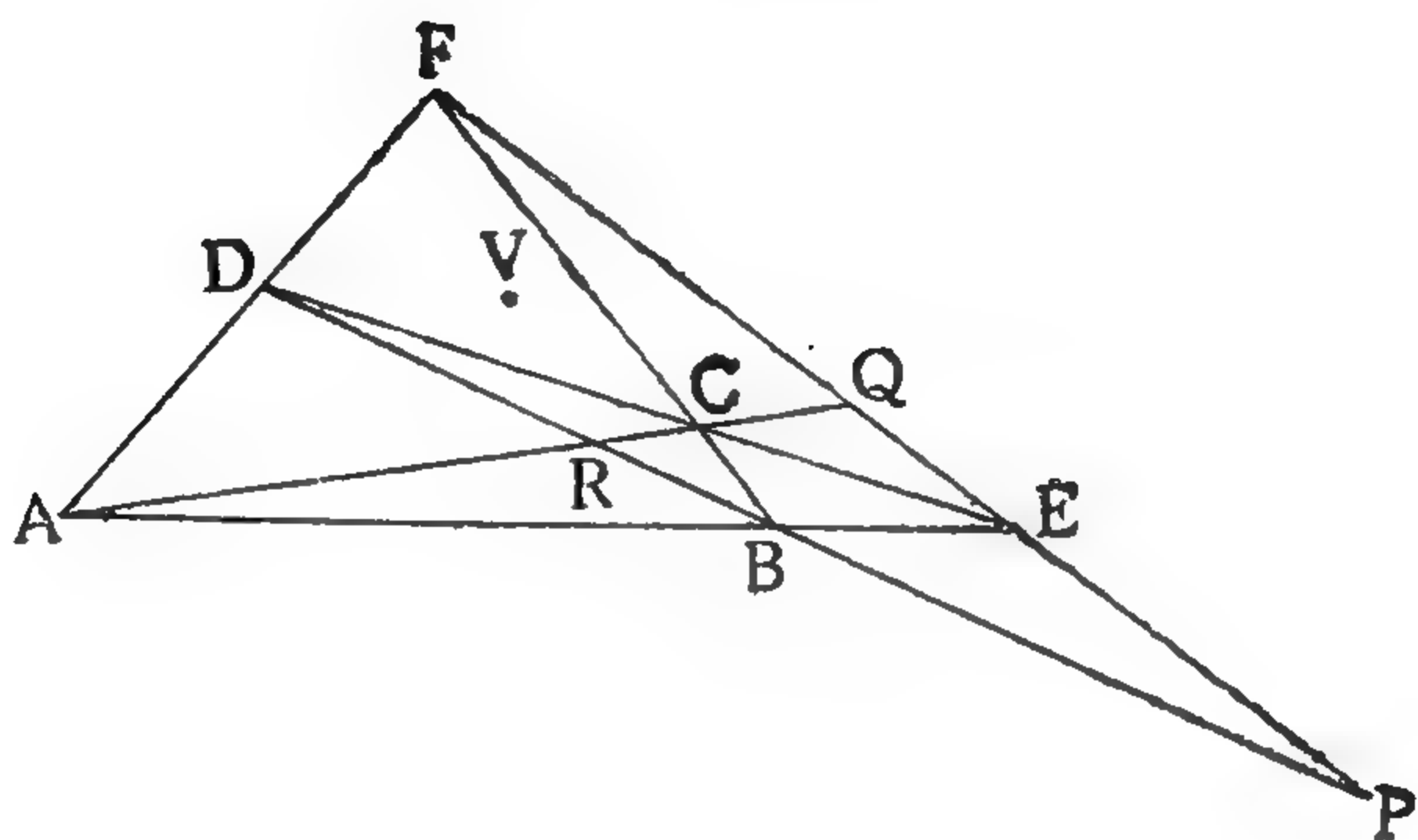


Fig. 76.

We have seen that $\mathcal{V}(A, C, B, D; E, F)$ is an involution pencil.

Since two pairs of corresponding rays $\mathcal{V}A, \mathcal{V}C$; $\mathcal{V}B, \mathcal{V}D$ are perpendicular, the involution is orthogonal.

$\therefore \mathcal{V}E, \mathcal{V}F$ are also perpendicular.

$\therefore$ the circle on EF as diameter passes through $\mathcal{V}$.

Similarly, it can be shown to pass through $\mathcal{V}'$.

Hence the three circles are co-axal.

Cor. 1. *The mid-points of the diagonals of a complete quadrilateral are collinear.*

Cor. 2. Since $(PQ, EF) = -1$, the circle drawn on EF as diameter cuts orthogonally every circle through P, Q and in particular, it cuts orthogonally the circumcircle of the triangle PQR . Similarly the circles drawn on AC, BD as diameters cut the circle through P, Q, R orthogonally. Hence the centre of circumcircle of $\triangle PQR$ must lie on the radical axis of the co-axal circles drawn on AC, BD, EF as diameters. Thus the radical axis of the three co-axal circles on the diagonals of a complete quadrilateral as diameters passes through the circum-centre of its diagonal triangle.

Ex. Show that the polar circles of the four triangles AED , BEC , AFB , CFD in fig. 76 belong to the co-axal system which is conjugate to the system of co-axals determined by the circles on the diagonals AC , BD , EF as diameters. Deduce that the ortho-centre of these four triangles are collinear.

Consider the triangle BEC , for example. BE is the polar of C for the polar circle of this triangle and A lies on BE , therefore A and C are conjugate points for the polar circle. Hence the circle on AC as diameter cuts the polar circle of $\triangle BEC$ orthogonally. Similarly, the circles on BD , EF cut the polar circle of BEC orthogonally.

Since the circles on AC , BD , EF as diameters are co-axal, the polar circle of BEC must belong to the conjugate system of co-axals, which cuts the circles of this system orthogonally.

Similarly for the polar circles of the remaining three triangles.

Since the four polar circles belong to a co-axal system, their centres viz., the ortho-centres of the four triangles are collinear.

8.7. To construct the pair of points harmonic w. r. t. each of two given pairs of collinear points.

Let A, B ; C, D be two given pairs of collinear points. We have to find two points P and Q so that

$$(AB, PQ) = -1, \text{ and } (CD, PQ) = -1$$

Draw circles S_1 and S_2 on AB and CD as diameters. Draw any other circle S meeting both these circles in real points.

Let the common chords of this circle and each of the two circles S_1 and S_2 meet in O' . Draw $O'O$ perpendicular to the line of centres $DCAB$ meeting it in O . From O draw a tangent OT to S_1 or S_2 . Set $OP = OQ = OT$ on the given line; then P and Q are the required points.

By construction, O' is the radical centre of S , S_1 and S_2 . The line $O'O$, being perpendicular to the line of centres of S_1 and S_2 is the radical axis of S_1 and S_2 .

Hence $OA.OB = OC.OD = OT^2$.

But $OT = OP = OQ$,

$\therefore OA.OB = OC.OD = OP^2 = OQ^2$.

$\therefore P, Q$ are harmonic w. r. t. each of the two pairs $A, B ; C, D$.

The student will see for himself that P, Q are the united points of the involution determined by the two given pairs of points. The construction fails if S_1 and S_2 intersect i. e. if AB and CD overlap. This is to be expected, for in this case, the involution is an overlapping one and as such the united points must be imaginary.

8.8. Co-axal circles and Projection. Since the circular points are common to every circle in a plane, a co-axal system of circles can be considered as having four points in common; two circular points and two other common points, real, coincident or imaginary. Hence a system of co-axal circles can be projected into a system of conics having four points in common. Such a system is called a *pencil of conics*.

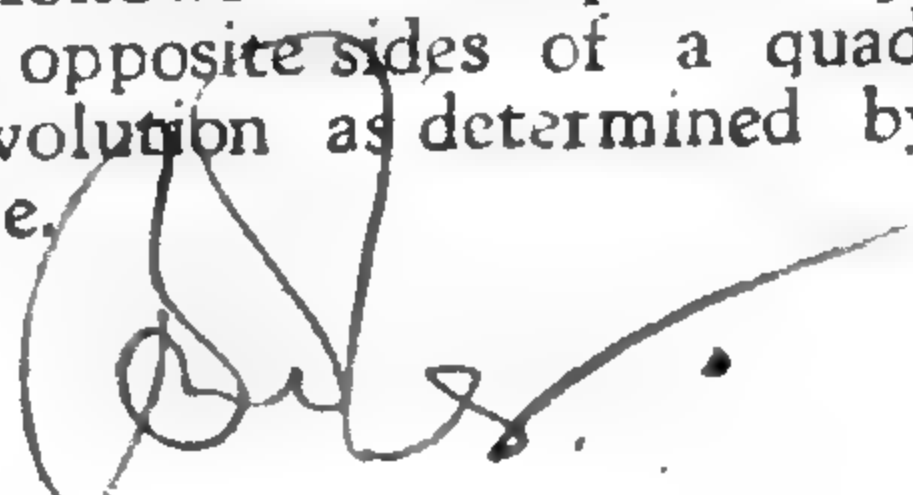
Note. If the four points are to be real, the vertex of projection must evidently be imaginary.

An important property of a pencil of conics can be deduced from that of a co-axal system of circles.

Theorem. *Every straight line cuts a pencil of conics in pairs of points which are in involution.*

Every straight line cuts a system of co-axal circles in pairs of points in involution. The result now follows if we note that a co-axal system of circles projects into a pencil of conics and a range in involution projects into a range in involution.

Cor. Since the opposite pairs of sides of the quadrangle formed by the four common points of the pencil of conics, can be looked upon as degenerate members of the system, it follows that the pairs of points in which a line cuts the opposite sides of a quadrangle belong to the same involution as determined by the pencil of conics on the line.



Exercises

1. Given four mutually orthogonal circles, show that their centres form the vertices of a triangle and its orthocentre.

2. AD, BE, CF are the altitudes of a triangle ABC and P is any point in its plane; prove that the circles circumscribing the triangles PAD, PBE, PCF are coaxal.

3. P, Q, R are points on the sides BC, CA, AB respectively of a triangle ABC ; prove that the radical centre of the circles described on AP, BQ and CR as diameters is the orthocentre of the $\triangle ABC$.

4. A system of circles have the same pair of points conjugate for each circle of the system, then the radical axes of the circles taken in pairs are concurrent

5. If a system of circles have a common pair of inverse points the system must be a coaxal one.

6. P is any point in the plane of a system of coaxal circles whose limiting points are L' and L'' ; show that the chord of contact of tangents drawn from P to any one of the circles all pass through the other extremity of the diameter through P of the circumcircle of the triangle $PL'L''$.

7. Three circles S_1, S_2, S_3 are such that the radical axis of S_1 and S_2 passes through the centre of S_3 and the radical axis of S_2 and S_3 passes through the centre of S_1 ; show that the radical axis of S_3 and S_1 passes through the centre of S_2 .

8. A circle can be drawn to cut three given circles orthogonally. If any point be taken on the circle its polars w. r. t. the three circles are concurrent.

9. If the polar of a point with respect to three circles whose centres are on a straight line, are concurrent, prove that the three circles are coaxal.

10. The tangents at A, B, C to the circumcircle of a triangle ABC cut BC, CA, AB at P, Q, R respectively. If U, V, W be the mid-points of AP, BQ, CR respectively, show that the radical axis of the circum-circle and the polar circle of the triangle ABC is the line UVW .

11. If AD, BE, CF are the altitudes of a triangle ABC and if

$$-1 = (BC, DP) = (CA, EQ) = (AB, FR);$$

show that PQR is the radical axis of the circumcircle and the nine-point circle [It is known that the points D, E, F , the three mid-points of the sides and the three midpoints of the lines joining the orthocentre to the vertices lie on a circle called *Nine-point circle*.]

12. AD, BE and CF are three concurrent lines drawn through the vertices of a triangle ABC ; EF meets BC at D_1 , FD meets CA at E_1 and DE meets AB at F_1 ; show that the circles on DD_1, EE_1 and FF_1 as diameters all cut the circumcircle of ABC orthogonally and are coaxal.

13. The three circles of similitude of three given circles taken in pairs are coaxal.

[The circle of similitude of two circles is the locus of a point which moves such that the ratio of its distances from the centres equals the ratio of their radii. The locus can easily be shown to be a circle]

14. Any line through the radical centre of three circles cut them in pairs of points in involution.

15. A line meets three circles in three pairs of points in involution; show that either the circles are co-axal or the line passes through their radical centre.

16. If three non concurrent lines meet three circles in pairs of points in involution, the circles are coaxal.

17. Show that the polars of a point P on the radical axis of two circles meet at another point Q on the radical axis and the pairs of points P, Q form an involution.

18. The six radical axes of four circles through the same point form an involution.

19. A variable circle passes through the fixed points A, B and cuts a fixed circle in P, Q ; show that the variable line PQ passes through a fixed point.

20. Show that the six radical axes of four circles taken in pairs form the six sides of a complete quadrilateral.

21. The six radical axes of the four circles which touch the sides of a triangle form a triangle and its orthocentre.

22. If A, B, C be the centres and a, b, c the radii of three coaxal circles, then

$$a^2 \cdot BC + b^2 \cdot CA + c^2 \cdot AB + BC \cdot CA \cdot AB = 0.$$

23. If t_1, t_2, t_3 are tangents from any point P to three coaxal circles whose centres are A, B, C , show that

$$t_1^2 \cdot BC + t_2^2 \cdot CA + t_3^2 \cdot AB = 0$$

HINTS TO EXERCISES

CHAPTER II

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2. Let the arms of two given angles meet in the points P, Q . Project the line PQ to infinity.

3. The line OPA and $O'PA'$ which intersect at P are coplanar and therefore the lines AA' and OO' are intersecting. The line AA' meets OO' at the fixed point H where OO' meets the plane π . Similarly BB' and CC' pass through H .

Axis of projection is the axis of perspective.

4. Let PQR and $P'Q'R'$ be any two positions of the variable triangle. These triangles are in perspective, the line ABC being the axis. The lines PP', QQ', RR' being concurrent, the line RR' passes through the point of intersection of PP', QQ' , which is a fixed point being the meet of the fixed lines on which P and Q lie.

6. Let the points of intersection of the pairs of sides AB, A_1B_1 , AC, A_1C_1 ; AD, A_1D_1 ; BC, B_1C_1 ; BD, B_1D_1 be collinear. We have to show that the lines CD, C_1D_1 meet on the same line.

By hypothesis, the pairs of triangles $ABC, A_1B_1C_1$ and $ABD, A_1B_1D_1$ are in perspective so that we deduce that the lines AA_1, CC_1 and DD_1 are concurrent. Thus the triangles ACD and $A_1C_1D_1$ are in perspective and the lines CD, C_1D_1 meet on the line joining the points of intersection of the pairs AC, A_1C_1 ; AD, A_1D_1 .

7. Let O be the centre of perspective of the triangles $ABC, A'B'C'$. Show that the triangles $OB'A'$ and $C'A_1B_1$ are in perspective and deduce that lines $AB, A'B', A_1B_1$ are concurrent.

8. Let the points I, J, K, L, M, N lie on the sides BD, AC, AD, BC, CD, AB respectively. Show that the triangles BDA, EFG are

in perspective so that the points I, K, V are collinear. It may similarly be shown that the sets of points $I, L, M; J, K, M; J, L, V$ are collinear.

9. Let $A_1B_1C_1, A_2B_2C_2, A_3B_3C_3$ be three given triangles. Let $P_1Q_1R_1; P_2Q_2R_2, P_3Q_3R_3$ be the three axes of perspective.

Since the three pairs of triangles have the same centre of perspective, the points $A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3$ are collinear, the lines being concurrent at the common centre. The triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are co-axial, axis being the line $C_1C_2C_3$. Therefore the lines P_1Q_1, P_2Q_2, P_3Q_3 are concurrent.

10. Let P, Q, R be the three meets. Project the line PQ to infinity. Denoting the projections of A, B, C, A', B', C' by $A_1, B_1, C_1; A'_1, B'_1, C'_1$ we see that $B'_1C_1 \parallel B_1C'_1, C'_1A_1 \parallel C_1A'_1$ and we have to show that $A_1B'_1 \parallel A'_1B_1$. Let the two lines $A_1B_1C_1$ and $A'_1B'_1C'_1$ concur at O_1 . We have

$$\frac{O_1B_1}{O_1C'_1} = \frac{O_1C_1}{O_1B'_1}, \quad \frac{O_1C'_1}{O_1A_1} = \frac{O_1A'_1}{O_1C_1}$$

$$\frac{O_1A_1}{O_1B'_1} = \frac{O_1B_1}{O_1A'_1}, \text{ i.e., } A_1B'_1 \parallel B_1A'_1.$$

11. Consider the two sets of collinear points ACB and $A'B'C'$ so that the meets of $AB', A'C; AC', A'B; CC', BB'$ are concurrent.

12. Let l be the axis of projection and β, γ the two planes bisecting the angles between the given planes α and π . From O draw perpendiculars to β, γ meeting α in M, N and π in M_1, N_1 . Then M, N are the required points. Take X, Y any two points on l and show that

$$\angle XMY = \angle XM'Y; \angle XNY = \angle XN'Y.$$

13. Let α, π be the given planes and P the given point on α . Let l' be any line in $\alpha \parallel$ the axis of projection l . Let π' be the plane through $l' \parallel \pi$. Draw $PX \perp l'$ and through X , lying in π' , draw $VXV', \perp l'$ and take $VX = PX = V'X$. Then V, V' are the required points. Take any angle with its vertex at P whose vanishing points are A and B . The triangles PXA and VXA being congruent, $\angle XPA = \angle XVA$; similarly $\angle XPB = \angle XVB$. Therefore $\angle APB = \angle AVB$.

15. Project two angles of either triangle into angles of 60° each and the axis of perspective to the line at infinity.

CHAPTER III

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1. $CC' = \frac{1}{2}(CA' + CB')$.

2. Insert origin at O .3. (a) Draw $AL \perp BC$.(b) Let AD be the internal bisector of the angle A so that

$$BD/DC = c/b.$$

i. e., $b.BD + c.CD = 0$. We have

$$b.PB^2 + c.PC^2 = b.BD^2 + c.CD^2 + (b+c)PD^2.$$

The in-centre I divides AD such that $a.AI + (b+c)DI = 0$. We have

$$a.PA^2 + (b+c)PD^2 = a.AI^2 + (b+c)DI^2 + (a+b+c)PI^2.$$

$$\therefore a.PA^2 + b.PB^2 + c.PC^2 = a.AI^2 + b.BD^2 + c.CD^2 + (b+c)DI^2 + (a+b+c)PI^2.$$

so that PI is constant.4. Insert origin at O .

5. $OA + AA' + A'O' + O'O = 0$.

6. $AA' + AB' + AC' + \text{etc.} = n'.OO' - n'.OA,$

7. Draw PL perpendicular to the line AB , L being the foot.

$$\begin{aligned} \text{L.H. S.} &= \sum BC(LP^2 + LA^2) = \sum BC.LA^2 + LP^2.(BC + CA + AB) \\ &= (AC - AB)LA^2 - AC.LB^2 + AB.LC^2, \\ &= AB.AC(LC + LA) - AC.AB(LA + LB), \\ &= AB.AC.BC. \end{aligned}$$

8. Let O , be the centre and r , the radius of the given circle; so that $l_1^2 = OA^2 - r^2$ etc.

CHAPTER III

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1. (iii) If AD be perpendicular to BC , $BD/CD = -c \cos B / b \cos C$.3. Since D, D', F, F' are concyclic, $BD.BD' = BF.BF'$

i. e., $BD/BF = BF'/BD'$.

Similarly write down the other similar results and multiply.

5. Let the tangent at A meet BC at D . The triangles ABD and ACD being similar,

$$\frac{AB}{AC} = \frac{BD}{AD} = \frac{AD}{CD} \text{ which gives } \frac{BD}{CD} = \frac{AB^2}{AC^2}.$$

6. Put down the conditions for the concurrency of the lines, AX , BY , CZ ; and the collinearity of the sets of points X, Y, Z' ; Y, Z, X' ; Z, X, Y' .

7. Let FF' , FD' , DE' meet BC , CA , AB in X , Y , Z respectively. Put down the conditions for the collinearity of the sets of points, D, E, F ; D', E', F' ; E, F', X ; F, D', Y ; D, E', Z .

8. Put down the condition for the collinearity of the points C, P, K lying on the sides of the triangle ABH .

10. Let CQ meet AB in L .

Put down the conditions for the collinearity of the points

(i) C, P, K lying on the sides of the $\triangle ABD$, this determines DP/AP .

(ii) B, Q, P „ „ $\triangle AHD$; „ HQ/AQ ;

(iii) C, Q, L „ „ $\triangle ABH$; „ AL/BL .

(iv) C, R, L „ „ $\triangle ABE$; „ BR/ER .

13. $\angle FEA = \angle B$ and $\angle EFA = \angle C$ so that $\frac{\sin \angle BAG}{\sin \angle CAG} = -\frac{\cos C}{\cos B}$; etc.

14. Let the pairs of corresponding sides of the two triangles meet in X, Y, Z . Now, put down the conditions for the transversals $B'C'$, $C'A'$, $A'B'$ of the triangle ABC .

15. Put down the conditions for the transversal AOU of $\triangle BYZ$, and for the transversal COZ of $\triangle ABY$; making use of the facts about parallelisms, we get the required relation.

16. Let EO meet AH at L , EO meet KD at M and EO meet BC at N .

The condition for the transversal COH of $\triangle ALE$ is

$$\frac{AH \cdot LO}{LH \cdot EO} \cdot \frac{EC}{AC} = 1 \text{ i. e., } \frac{BH \cdot LO}{EH \cdot EO} \cdot \frac{EC}{AC} = 1. \quad (1)$$

The condition for the transversal BOK of $\triangle DME$, similarly, is

$$\frac{CK \cdot MO}{EK \cdot EO} \cdot \frac{EB}{DB} = 1. \quad (2)$$

Also, since the lines CH , BK , EN concur at O ,

$$\frac{EH \cdot BN}{BH \cdot CN} \cdot \frac{CK}{EK} = -1, \quad (3)$$

From (1), (2) and (3), we get

$$\frac{LO \cdot EC \cdot DB \cdot BN}{AC \cdot MO \cdot EB \cdot CN} = -1. \quad (4)$$

From parallelisms, $\frac{EC}{AC} = \frac{CN}{CB} ; \frac{DB}{EB} = \frac{BN}{CN}$

so that (4) becomes

$\frac{LO}{MO} \cdot \frac{CN}{CB} \cdot \frac{BC}{BN} \cdot \frac{BN}{CN} = -1$ i.e., $LO = MO$ so that L and M coincide.

CHAPTER V

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$$2. \quad 2 \frac{PC}{AC} = \frac{PC}{AB} + \frac{PC}{AD} = \frac{BC - BP}{AB} + \frac{DC - DP}{AD} \text{ etc.,}$$

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1. If O be the point of intersection of the diagonals,

$$OP \cdot OQ = OA^2 = OB^2 = OX \cdot OY.$$

2. Since D, E, F are collinear, we have

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1. \quad \dots (1)$$

Also

$$\frac{BD}{CD} = -\frac{BD'}{CD'} \text{ etc., so that (1) becomes}$$

$$\frac{BD'}{CD'} \cdot \frac{CE'}{AE'} \cdot \frac{AF'}{BF'} = -1.$$

Hence the result.

3. AQ, BE, CF being concurrent,

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1; \quad \dots (1)$$

E, F, D' being collinear,

$$\frac{BD'}{CD'} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1. \quad \dots (2)$$

From (1) and (2), $BD/CD = -BD'/CD'$ so that $(BC, DD') = -1$.

4. Let AO, BO, CO meet BC, CA, AB in D, E, F and let EF meet BC in A' . Then, $A'O$ is the required line.

$$5. \quad \frac{BD'}{CD'} = -\frac{BD}{CD} = -\frac{PB}{PC} \text{ and so on.}$$

7. $\therefore (AB, C'D') = -1, \therefore (AB, D'C') = -1$. Thus $(ACBD) = (AD'B'C')$. Let CD', BB' meet in O ; show that OC' passes through D .

8. Take D such that $(AB, CD) = -1$ and project D to infinity.

10. Let AD, BE, CF be three concurrent lines.

Take D_1, E_1, F_1 , such that

$$-1 = (BC, DD_1) = (CA, EE_1) = (AB, FF_1).$$

Show that D_1, E_1, F_1 are collinear and project this line to infinity.

12. Let $x_1, x_2; x_3, x_4$ be the roots so that

$$x_1 + x_2 = -2b/a, x_1 x_2 = c/a; x_3 + x_4 = -2b'/a', x_3 x_4 = c'/a'.$$

The two point-pairs being harmonic,

$$(x_1 + x_2)(x_3 + x_4) = 2(x_1 x_2 + x_3 x_4)$$

$$\therefore ac' + a'c = 2bb'.$$

13. To prove (i) insert origin at O .

For (ii), we have

$$AC.BD = -AD.BC = -(CD - CA)(OC - OB).$$

Now, apply (i)

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1. Let C be the meet of AB and l . Then p always passes through D which is such that $(AB, CD) = -1$.

2. If D be the harmonic conjugate of B w.r.t. AC , then the locus is the circle described on BD as diameter.

3. Let FE and $F'E'$ meet BC at X and Y respectively. We have

$$\frac{BX}{CX} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1, \frac{BY}{CY} \cdot \frac{CE'}{AE'} \cdot \frac{AF'}{BF'} = 1.$$

Also $\frac{CE}{AE} = -\frac{CE'}{AE'}, \frac{AF}{BF} = -\frac{AF'}{BF'}$. From these we obtain

$$\frac{BX}{CX} = \frac{BY}{CY} \text{ so that } X, Y \text{ coincide.}$$

4. Let AB meet l and LM in U and V respectively; U being fixed. Considering the quadrangle ABP_1P_2 , we have $M(AB, LU) = -1$

$\therefore (AB, UV) = -1$ so that V is a fixed point.

5. The diagonal CD of the quadrilateral formed by the lines

CH, HD, DG, GC is cut harmonically by the other two diagonals at A and R and A, C, D being fixed, R is also fixed.

6. Let SU and RV meet BA at L and M . We have

$$-1 = (BC, UV) = S(BC, UV) = (BA, LP)$$

and $-1 = (BC, UV) = R(BC, UV) = (BA, PM) = (\cancel{BA, PM}) = (BA, MP)$

The points L, M being the harmonic conjugates P w. r. t. AB , must coincide.

Or

Let SU meet AB at L and let RL meet BC at U' . Considering the quadrangle $PQUU'$, show that the $(BC, UU') = -1$

7. Let AB, CD meet at P and BC, AD at Q .

Consider the quadrangle $ABCD$, so that we have $P(BC, OQ) = -1$. Considering its section by the transversal OGH , we get the required result.

$$-1 = (BC, OQ) = U(OG, UV) = (UV, GB) = (UG, VF)$$

8. $-1 = (UGVF) = B(UGVF) = (ACVL) = (ALVC)$ so that $(UGVF) = (ALVC)$; hence AU, LG, CF are concurrent.

10. Let PV, PR, UR meet AC at L, M, N .

$$-1 = (CV, RQ) = P(CV, RQ) = (CL, MA)$$

Also $-1 = (CU, PS) = R(CU, PS) = (CN, MA)$

$\therefore L, N$ coincide.

CHAPTER VI

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1. Project one of the two double points to infinity.

3. Follows from the equality $(U_2 U_1 PP') = (U_2 U_1 QQ')$.

4. Follows from the equalities

$$(AA' U_1 U_2) = -1 = (BB' U_2 U_1) \text{ and } (AA' U_1 U_2) = -1 = (B' B U_2 U_1).$$

5. Let AA' meet BC at L and let O be the centre of perspective. We have

$$(PQRP') = A(PQRP') = (PCBL)$$

$$(P'Q'R'P) = O(P'Q'R'P) = (LBCP) = (PCBL).$$

6. Let VA meet BC at A' and let $V\alpha, V\beta, V\gamma$ meet BC at α, β, γ so that α is a point at infinity. Considering the transversal BC , we have to show that

$$(A'BC\alpha_\infty) = (\alpha_\infty\beta\gamma A') \text{ i. e., } A'B/CB = \gamma A'/\gamma\beta.$$

From the theory of similar triangles,

$$\frac{A'B}{\gamma A'} = \frac{AB}{V\gamma} = \frac{CB}{\gamma\beta} \text{ i.e., } \frac{A'B}{CB} = \frac{\gamma A'}{\gamma\beta}.$$

7. If a transversal cuts the pairs of opposite sides in $P, P_1; Q, Q_1; R, R_1$; and O is any point, then $O(P, P_1; Q, Q_1; R, R_1)$ is an involution pencil. In case the transversal goes to infinity, the members of this pencil become parallel to the sides of the quadrangle..

8. Let U_1, U_2 be the double points. Project one of the two double points, say U_2 , to infinity so that the other U_1 , projects into the common mid-point of the segments determined by the projections of the point pairs. Let O be this common mid-point. We denote the projections by attaching a suffix 1. We have

$$OH_1 \cdot OP_1 = OA_1^2 \text{ etc.,}$$

$$\begin{aligned} (PQRS) &= (P_1Q_1R_1S_1) = \frac{(OQ_1 - OP_1)(OS_1 - OR_1)}{(OS_1 - OP_1)(OQ_1 - OR_1)} \\ &= \frac{(OB_1^2 - OA_1^2)(OD_1^2 - OC_1^2)}{(OD_1^2 - OA_1^2)(OB_1^2 - OC_1^2)} \\ &= \frac{(OB_1 - OA_1)(OD_1 - OC_1)}{(OD_1 - OA_1)(OB_1 - OC_1)} \cdot \frac{(OB_1/OA_1 + 1)(OD_1/OC_1 + 1)}{(OD_1/OA_1 + 1)(OB_1/OC_1 + 1)} \\ &= (A_1B_1C_1D_1) \cdot \frac{(B_1OA_1\alpha)(D_1OC_1\alpha)}{(D_1OA_1\alpha)(B_1OC_1\alpha)} \text{ where } \alpha \text{ denotes the point} \end{aligned}$$

at infinity.

$$= (ABCD) \cdot \frac{(BU_1AU_2)(DU_1CU_2)}{(DU_1AU_2)(BU_1CU_2)}.$$

Hence $(BAU_1U_2) = -1$, $\therefore (BU_1AU_2) = 1 - (-1) = 2$.

Similarly for others. Hence

$$(PQRS) = (ABCD).$$

11. $P, P'; Q, Q'; R, R'$ are in involution.

12. Taking the centre O of the involution as origin, the analytical relation which determines the involution takes the form $xx' = k$. Let $OA = x_1, OA' = x'_1 = x_1/k, OB = x_2; OB' = x'_2 = x_2/k$ etc.

$$\therefore OP = (OA + OA')/2 = (x_1 + x'_1)/2 = (x_1 + k/x_1)/2 = (x_1^2 + k)/2x_1.$$

Similarly $OQ = (x_2^2 + k)/2x_2$ etc.

Now compute the three cross ratios.

13. Let P be a point of intersection of the circles drawn on AA', BB' as diameters. The required point C' is the point where the line through P drawn at right angles to PC meets the given line. Draw

$PO \perp$ the given line. We have $OA.OA' = OB.OB' = OC.OC'$, each of them being equal to OP^2 .

CHAPTER VII

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1. The fixed line is the polar of the fixed point.
2. The fixed point is the pole of the fixed line.
3. If AB cuts the circle in C, D , then $PT^2 = FC.PD = PA^2$. (ACBD) = P
Lose P
4. The point at infinity on the chord bisected at P , being conjugate to P , lies also on the line QR .

5. S, P and S, T are two pairs of conjugate points and hence TP is the polar of S .

Let TP meet AB at C .

$$\therefore (PS, QR) = T(PS, QR) = (CS, AB) = -1.$$

6. Follows from above.

7. If C be the centre, the pole A of l lies on OC and APQ is a self-conjugate triangle whose ortho-centre is C . The triangles OAP, OCQ being similar,

$$\begin{aligned} PO.OQ &= OC.OA = OC(OC - AC) = OC^2 - OC.AC \\ &= OC^2 - (\text{radius})^2 = (\text{tangent})^2. \end{aligned}$$

8. Show that the poles of any chord through a centre of similitude S of two circles are collinear with S .

9. Let VK be the polar of V' meeting PP' in K . We have $(V'K, PP') = -1$ and $\therefore V(V'K, PP') = -1$.

VV' and VK being fixed lines, VP and VP' generate an involution pencil; if $V'T, V'T'$ be tangents through V' and T, T' ; their points of contact then VT, VT' are the two double lines.

11. The points H, B are the centres of the polar circles of the triangles ABC and ACH . Let HB meet AC at L .

$HB.HL$ = square of the radius of the polar circle of $\triangle ABC$.

$BL.BH$ = " " " " $\triangle ACH$.

$\therefore HB.HL + BL.BH = HB^2$ = square of the distance between their centres. Hence the result.

1. Since the centre of a circle which cuts two given circles orthogonally lies on their radical axis, the line joining the centres of any two circles, being the radical axis of the other two, is perpendicular to the line joining their centres.

2. If H , be the orthocentre,

$$HA.HD=HB.HE=HC.HF$$

so that the power of H w.r. to the three circles are equal. Thus the radical axis of each pair passes through the same two points H and P .

4. If the line joining the conjugate points U', U'' meet the system of circles in point pairs A_1, A_2 ; B_1, B_2 ; C_1, C_2 ; etc.,

$$-1=(U'U'', A_1A_2)=(U'U'', B_1B_2)=(U'U'', C_1C_2)$$

so that U', U'' are the double points of the involution determined by the point pairs A_1, A_2 ; B_1, B_2 The power of the mid-point O of $U'U''$ is the same for every circle of the system.

6. The circle $P L'L''$ cuts every member of the co-axial system orthogonally. If the diameter of the circle $P L'L''$ drawn through P meets any circle in A_1, A_2 and Q , be the other extremity of the diameter,

$$(PQ, A_1A_2)=-1$$

so that P and Q are conjugate points for every circle of the system.

7. Let C_1, C_2, C_3 be the three centres. The radical axis of S_1 and S_2 is the line through C_3 perp. to C_1C_2 and the radical axis S_2 and S_3 is the line through C_1 perp. to C_2C_3 . The radical centre of the three circles, is therefore, the orthocentre of the $\triangle C_1C_2C_3$. Hence the radical axis of S_3S_1 passes through the orthocentre and is perp. to C_1C_3 and therefore passes through C_2 .

8. If P be any point on the orthogonal circle, its polars w.r. to the three circles concur at the other extremity of the diameter through P of the orthogonal circle,

9. Let the polars of A concur at B . Then the powers of the mid-point O of AB w.r. to the three circles are equal.

10. The points A, P being conjugate for the polar circle, the power of U w. r. t. it is UA^2 which is also its power w. r. t. the circumcircle.

11. If L be the mid-point of BC , we have $2/PD = 1/PB + 1/PC$
i. e. $PB \cdot PC = (PB + PC)PD/2 = PL \cdot PD$

so that the powers of P w. r. t. the two circles are equal.

13. Let C_1, C_2, C_3 be the centres and r_1, r_2, r_3 the radii of the three circles S_1, S_2, S_3 . If P be a point of intersection of the circle of similitude of S_1, S_2 and S_2, S_3 , we have

$$\frac{PC_1}{PC_2} = \frac{r_1}{r_2}; \quad \frac{PC_2}{PC_3} = \frac{r_2}{r_3}$$

$$\therefore \frac{PC_1}{PC_3} = \frac{r_1}{r_3}$$

so that P lies on the circle of similitude of S_1, S_3 , also, and so on

14. Let any line through the radical centre O cut the three circles in point pairs A_1, A_2 ; B_1, B_2 ; C_1, C_2 . The power of O , w. r. t. the three circles being the same,

$$OA_1 \cdot OA_2 = OB_1 \cdot OB_2 = OC_1 \cdot OC_2$$

17. Let the radical axis of the two circles meet them in A and B . If the polar of P w. r. t. one circle meets the radical axis in Q , $(PQ, AB) = -1$ so that P, Q are conjugate w. r. t. the second circle also. Since the point pairs P, Q , are harmonically conjugate w. r. t. a fixed pair of points A, B they form an involution.

18. Consider the quadrangle formed by the centres of the circles and compare with Ex. 7. Page 96. Ch. VI.

20. Since $4C_3 = 4$, the four radical centres of the four circles determine the vertices of a complete quadrangle whose six sides are the six radical axes.

19. Let S_1, S_2, S_3 etc., be different positions of the variable circle and let S be the fixed circle. Let S meet S_1, S_2, S_3 etc. in lines P_1Q_1, P_2Q_2, P_3Q_3 etc.

Let P_1Q_1 meet at AB at O . Now, AB is the radical axis of S_1, S_2 and P_1Q_1 of S_1, S . $\therefore O$ is the radical centre of S, S_1, S_2 so that the radical axis P_2Q_2 of S, S_2 passes through O . Similarly P_3Q_3, P_4Q_4 etc, all pass through O .

22. Let S be any point on the radical axis and t the tangent from S to either circle. We have

$$a^2 = PA^2 - t^2 \text{ etc.}$$

$$\therefore L.H.S. = PA^2.BC + PB^2.CA + PC^2.AB + BC.CA.AB = 0$$

(Refer to Ex. 7, Ch. III Page 33)

23. $t_1^2 = SA^2 - a^2$ etc,

$$\begin{aligned} \therefore L.H.S. &= PA^2.BC + PB^2.CA + PC^2.AB - (a^2.BC + b^2.CA + c^2.AB) \\ &= -BC.CA.AB + BC.CA.AB = 0. \end{aligned}$$

THE END

